

# APPROXIMATE AND LOW REGULARITY DIRICHLET BOUNDARY CONDITIONS IN THE GENERALIZED FINITE ELEMENT METHOD

IVO BABUŠKA, VICTOR NISTOR, AND NICOLAE TARFULEA

**ABSTRACT.** We propose a method for treating the Dirichlet boundary conditions in the framework of the Generalized Finite Element Method (GFEM). We are especially interested in boundary data with low regularity (possibly a distribution). We use approximate Dirichlet boundary conditions as in [11] and polynomial approximations of the boundary. Our sequence of GFEM-spaces considered,  $S_\mu$ ,  $\mu = 1, 2, \dots$  is such that  $S_\mu \not\subset H_0^1(\Omega)$ , and hence it does not conform to one of the basic FEM conditions. Let  $h_\mu$  be the typical size of the elements defining  $S_\mu$  and let  $u \in H^{m+1}(\Omega)$  be the solution of the Poisson problem  $-\Delta u = f$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , on a smooth, bounded domain  $\Omega$ . Assume that  $\|v_\mu\|_{H^{1/2}(\partial\Omega)} \leq Ch_\mu^m \|v_\mu\|_{H^1(\Omega)}$  for all  $v_\mu \in S_\mu$  and  $|u - u_I|_{H^1(\Omega)} \leq Ch_\mu^m \|u\|_{H^{m+1}(\Omega)}$ ,  $u \in H^{m+1}(\Omega) \cap H_0^1(\Omega)$ , for a suitable  $u_I \in S_\mu$ . Then we prove that we obtain quasi-optimal rates of convergence for the sequence  $u_\mu \in S_\mu$  of GFEM approximations of  $u$ , that is,  $\|u - u_\mu\|_{H^1(\Omega)} \leq Ch_\mu^m \|f\|_{H^{m-1}(\Omega)}$ . We also extend our results to the inhomogeneous Dirichlet boundary value problem  $-\Delta u = f$  in  $\Omega$ ,  $u = g$  on  $\partial\Omega$ , including the case when  $f = 0$  and  $g$  has low regularity (*i.e.*, is a distribution). Finally, we indicate an effective technique for constructing sequences of GFEM-spaces satisfying our conditions using polynomial approximations of the boundary.

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## INTRODUCTION

In the past few years, meshless methods for the approximation of solutions of partial differential equations have received increasing attention, especially in the Engineering and Physics communities. The reasons behind the development of such methods are the difficulties associated to the mesh generation, particularly when the geometry of the domain is complicated. As in the case of the usual Finite Element Method, one of the major problems in the implementation of meshless methods is the enforcement of Dirichlet boundary conditions. It is the purpose of this paper to address the problem of enforcing Dirichlet boundary conditions in the Generalized Finite Element Method framework. We are especially interested in the case when the Dirichlet boundary data has low regularity, including the case when it is a distribution on the boundary.

The classical Rayleigh-Ritz methods for elliptic Dirichlet boundary value problems assume that the trial subspace functions fulfill the boundary conditions. Nevertheless, the construction of such subspaces implies many difficulties in practice when the boundary of the domain is curved. Therefore, several approaches are known for dealing with the Dirichlet boundary conditions on domains with curved boundaries. One approach is to modify the variational principles by adding appropriate boundary terms so that there will be no need for the trial subspaces to fulfill any condition at the boundary. See the works of Babuška [2, 4], Bramble and Nitsche [12], and Bramble and Schatz [13, 14], among others, for examples of how this approach works in practice. Another approach (used also in this paper) is to use subspaces with nearly zero boundary conditions. This idea was first outlined by Nitsche [27] and further studied by Berger, Scott, and Strang [11] and Nitsche [28].

Yet another approach is the Isoparametric Finite Element Method or IFEM with *curved* finite elements along the boundary. See [18] and references therein, or [17], [19, 21, 23, 24, 32, 33], among many others, for more recent work and applications. This approach is typically used in connection with a numerical quadrature scheme computing the coefficients of the resulting linear systems. In the applications of this method, except in special cases (such as when  $\bar{\Omega}$  is a polyhedral domain) the interior  $\Omega_h$  of the union of the finite elements is not equal to  $\Omega$ , although the boundary of  $\Omega_h$  is very close to  $\partial\Omega$ . That is, the approximate solution  $u_h$  is sought in a subspace  $V_h \subset H_0^1(\Omega_h)$  and so, the homogeneous Dirichlet boundary condition  $u = 0$  on  $\partial\Omega$  is “approximated” by the boundary condition  $u_h = 0$  on  $\partial\Omega_h$ . In fact,  $u_h$  is the solution of a variational equation  $a_h(u_h, v_h) = (f_h, v_h)_h$  for all  $v_h \in V_h$ , where  $a_h(\cdot, \cdot)$  is a bilinear form which approximates the usual bilinear form defined over  $H^1(\Omega_h) \times H^1(\Omega_h)$ , and  $f_h \in V_h^*$  approximates the linear form  $v_h \in V_h \rightarrow \int_{\Omega_h} \tilde{f} v_h dx$ , where  $\tilde{f}$  is an extension of  $f$  to the set  $\Omega_h$ .

Our approach has certain points in common with the isoparametric method just mentioned in the fact that we are using polynomial approximations of the boundary. However, our method does not require non-linear changes of coordinates. Our method combines the approaches in the papers of Berger, Scott, and Strang [11]

and Nitsche [28]. Our definition of the discrete solution is as in [11], whereas our assumptions are closer to those of [28]. We have tried to keep our assumptions at a minimum. This is possible using partitions of unity, more precisely the Generalized Finite Element Method or GFEM, a method that originated in the work of Babuška, Caloz, and Osborn [6] and further developed in [4, 7, 8, 10, 22, 25, 26, 34].

Our construction is different from the IFEM in that we do not require complicated non-linear changes of coordinates. Moreover, our method uses non-conforming subspaces of functions and it does not have to deal with extensions over larger domains. It is closely related to [9] which uses GFEM for elliptic Neumann boundary value problems with distributional boundary data. The GFEM is a generalization of the meshless methods which use the idea of partition of unity. This method allows a great flexibility in constructing the trial spaces, permits inclusion of a priori knowledge about the differential equation in the trial spaces, and gives the option of constructing trial spaces of any desired regularity. We mention that the GFEM is also known and used under other names, such as: the method of “clouds,” the method of “finite spheres,” the “X-finite element method,” and others. See the survey by Babuška, Banerjee, and Osborn [4] for further information and references.

Let us now describe the main results of this paper in some detail. Let  $\Omega \subset \mathbb{R}^n$  be a *smooth, bounded* domain with boundary  $\partial\Omega$ . Let  $f \in H^{-1}(\Omega)$  and  $u \in H^1(\Omega)$  be the unique solution of the Poisson problem

$$(1) \quad -\Delta u = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

(For most of our results, we shall assume that  $f \in H^{p-1}(\Omega)$  with  $p \geq 1$ .)

Assume that we are given a sequence  $h_\mu \rightarrow 0$  and a sequence  $S_\mu \subset H^1(\Omega)$  of test-trial spaces. The parameters  $h_\mu$  play the role of the size of the elements defining  $S_\mu$ . We define the discrete solution  $u_\mu \in S_\mu$  in the usual way:  $B(u_\mu, v_\mu) = (f, v_\mu)$  for all  $v_\mu \in S_\mu$ , where  $B(u, v) := \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$  (see Equation (3) below). We do not assume  $S_\mu$  to satisfy exactly the Dirichlet boundary conditions, that is, we do not assume  $S_\mu \subset H_0^1(\Omega)$ .

Let us fix from now on a natural number  $m \in \mathbb{N} = \{1, 2, \dots\}$  that will play, in what follows, the role of the expected *order of approximation*. We shall make the following two basic assumptions. The first assumption is that our approximating functions satisfy Dirichlet boundary conditions approximately:

- *Assumption 1, nearly zero boundary values:*  $\|v_\mu\|_{H^{1/2}(\partial\Omega)} \leq Ch_\mu^m \|v_\mu\|_{H^1(\Omega)}$  for any  $v_\mu \in S_\mu$ .

The second assumption is an approximation assumption that will be used also for non-homogeneous boundary conditions. For that purpose, let us consider a second sequence of subspaces  $\tilde{S}_\mu \subset H^1(\Omega)$ ,  $S_\mu \subset \tilde{S}_\mu$ , which are not required to satisfy any boundary conditions.

- *Assumption 2, approximability:* for any  $u \in H^{j+1}(\Omega)$ ,  $0 \leq j \leq m$ , then there exists  $u_I \in \tilde{S}_\mu$  such that  $|u - u_I|_{H^1(\Omega)} \leq Ch_\mu^j \|u\|_{H^{j+1}(\Omega)}$ . If  $u = 0$  on  $\partial\Omega$ , then we can take  $u_I \in S_\mu$ .

These two assumptions are formulated in more detail in Section 1.

Our paper is divided into two parts. In the first part, consisting of the first four sections, we prove some general approximation results for the Poisson problem with Dirichlet boundary conditions. The approximations (or discrete solutions) belong to some abstract spaces  $S_\mu$  (for zero boundary conditions) or  $\tilde{S}_\mu$  (for general boundary conditions) that are required to satisfy certain reasonable assumptions

(Assumptions 1 and 2, for zero Dirichlet boundary conditions, and Assumptions 1, 2, and 3 for non-zero boundary conditions). In the second part of the paper we provide examples of Generalized Finite Element Spaces satisfying these assumptions. Under Assumptions 1 and 2, our main approximation result in Section 1 is the following.

**Theorem 0.1.** *Let  $S_\mu \subset H^1(\Omega)$  be a sequence of finite dimensional subspaces satisfying Assumptions 1 and 2 for a sequence  $h_\mu \rightarrow 0$  and  $0 \leq p \leq m$ . Then the (unique) solutions  $u$  and  $u_\mu$  of Equations (1) and (3), respectively, with  $f \in H^{p-1}(\Omega)$  satisfy*

$$\|u - u_\mu\|_{H^1(\Omega)} \leq Ch_\mu^p \|u\|_{H^{p+1}(\Omega)} \leq Ch_\mu^p \|f\|_{H^{p-1}(\Omega)},$$

for constants independent of  $\mu$  and  $f \in H^{p-1}(\Omega)$ .

In Section 2 we extend our results to the non-homogeneous Dirichlet boundary conditions case  $u = g$  on  $\partial\Omega$ , with  $g \in H^{m+1/2}(\partial\Omega)$ . In essence, we will be looking for a sequence  $G_k$  of *approximate extensions* of  $g$ , that is, a sequence of elements of  $H^{m+1}(\Omega)$  subject to the following assumption. Recall that the sequence  $h_\mu$  should be thought of as the “typical size” of the elements defining  $S_\mu$  and satisfies  $h_\mu \rightarrow 0$ .

- *Assumption 3, approximate extensions:* There exists a constant  $C > 0$  such that, for any  $g \in H^{m+1/2}(\partial\Omega)$ , there exists a sequence  $G_k \in \tilde{S}_k$  such that  $\|G_k|_{\partial\Omega} - g\|_{H^{1/2}(\partial\Omega)} \leq Ch_k^m \|g\|_{H^{m+1/2}(\partial\Omega)}$  and  $\|G_k\|_{H^{m+1}(\Omega)} \leq C\|g\|_{H^{m+1/2}(\partial\Omega)}$ .

Let  $w_k$  be the exact solution of  $-\Delta w_k = f + \Delta G_k$  in  $\Omega$ ,  $w_k = 0$  on  $\partial\Omega$ . Also, let  $(w_k)_\mu \in S_\mu$  be the discrete solution of this equation, namely, the solution of the discrete variational problem

$$(2) \quad B((w_k)_\mu, v) = \langle f + \Delta G_k, v \rangle_{L^2(\Omega)}, \quad v \in S_\mu,$$

where  $f \in H^{m-1}(\Omega)$  is the data of Equation (13). The result we prove in Section 2 is the following.

**Theorem 0.2.** *Suppose Assumptions 1, 2, and 3 are satisfied. Let  $u_k := (w_k)_k + G_k$ . Then there exists a constant  $C > 0$  such that the solution  $u \in H^{m+1}(\Omega)$  of Equation (13) satisfies*

$$\|u - u_k\|_{H^1(\Omega)} \leq Ch_k^m (\|f\|_{H^{m-1}(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)}).$$

In order to deal with low regularity boundary data, in Section 3 we consider the Dirichlet problem  $-\Delta u = f$  in  $\Omega$ ,  $u = g$  on  $\partial\Omega$ , with  $g \in H^{1/2-s}(\partial\Omega)$  and  $f \in H^{-1-s}(\Omega)$ ,  $s > 0$ . Thus both  $f$  and  $g$  may be distributions. We say that  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega) := H^{1-s}(\Omega) \oplus H^{-1/2-s}(\partial\Omega)$  is a *weak solution* of the problem  $-\Delta u = f$  in  $\Omega$ ,  $u = g$  on the boundary  $\partial\Omega$  if  $\tilde{B}(u, v) = -\langle f, v \rangle_\Omega + \langle g, \partial_\nu v \rangle_{\partial\Omega}$ , for all  $v \in H^{1+s}(\Omega)$ . Then, the result of Section 3 is the following.

**Theorem 0.3.** *Let  $g \in H^{1/2-s}(\partial\Omega)$  and  $f \in H^{-1-s}(\Omega)$ . Then there exists a unique weak solution  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$  for the problem  $-\Delta u = f$  in  $\Omega$ ,  $u = g$  on the boundary  $\partial\Omega$  and this solution satisfies*

$$\|u_0\|_{H^{1-s}(\Omega)} + \|u_1\|_{H^{-1/2-s}(\partial\Omega)} \leq C_{\Omega,s} (\|g\|_{H^{1/2-s}(\partial\Omega)} + \|f\|_{H^{-1-s}(\Omega)}),$$

for a constant  $C_{\Omega,s}$  that depends only on  $\Omega$  and  $s$ .

The second part of this paper is dedicated to constructing concrete examples of Generalized Finite Element Spaces  $S_\mu$  and  $\tilde{S}_\mu$  satisfying the Assumptions 1, 2, and 3 of the first part. In fact, we will prove that these assumptions are easy to fulfill with a “flat-top” partition of unity and polynomial local approximation spaces. The exact conditions are formulated in Section 4. The proof that the resulting GFEM spaces satisfy the Assumptions 1, 2, and 3 is in Section 5. In addition, in Section 6 we also prove interior estimates for the error  $u - u_k$ , where  $u$  is the solution of the distributional boundary value problem  $-\Delta u = 0$  in  $\Omega$ ,  $u = g$  on  $\partial\Omega$ , with distributional data  $g \in H^{1/2-s}(\partial\Omega)$ ,  $s > 0$ , and  $u_k \in \tilde{S}_k$  is the discrete solution in our GFEM spaces. The last section contains some comments and a discussion of some further problems.

In this paper, we shall use the convention that  $C > 0$  indicates a generic constant, independent of  $\mu$ , which may be different each time when used, but is independent of the free variables of the formulas.

## Part 1. Approximate Dirichlet boundary conditions

### 1. HOMOGENEOUS BOUNDARY CONDITIONS

In this section, we give a proof of Theorem 0.1. We begin by fixing the notation and then we prove some preliminary results. In particular,  $f \in H^{-1}(\Omega)$  and  $u \in H_0^1(\Omega)$  is the solution of the Poisson problem (1).

Recall that  $\Omega \subset \mathbb{R}^n$  is a *smooth, bounded domain*, fixed throughout this paper. We shall fix in what follows  $m \in \mathbb{N} = \{1, 2, \dots\}$ , which will play the role of *the order of approximation*. We want to approximate  $u$  with functions  $u_\mu \in S_\mu$ ,  $\mu \in \mathbb{N}$ , where  $S_\mu \subset H^1(\Omega)$  is a sequence of finite dimensional subspaces that satisfy the Assumption 1 and 2 formulated next. In those assumptions, the sequence  $h_\mu \rightarrow 0$  should be thought of as the “typical size” of the elements defining  $S_\mu$ . Our first assumption is:

- **Assumption 1 (nearly zero boundary values).** There exists  $C > 0$  such that

$$\|v_\mu\|_{H^{1/2}(\partial\Omega)} \leq Ch_\mu^m \|v_\mu\|_{H^1(\Omega)} \quad \text{for any } v_\mu \in S_\mu.$$

So  $S_\mu$  *does not* necessarily consist of functions satisfying the Dirichlet boundary conditions. Let  $|u|_{H^1(\Omega)} := [\int_\Omega |\nabla u|^2 dx]^{1/2}$ . Our second assumption will also be used for non-homogeneous boundary conditions, so we formulate it also for a sequence of spaces  $\tilde{S}_\mu \subset H^1(\Omega)$ ,  $S_\mu \subset \tilde{S}_\mu$ .

- **Assumption 2 (approximability):** There exists  $C > 0$  such that for any  $0 \leq j \leq m$ , any  $u \in H^{j+1}(\Omega)$ , and any  $\mu \in \mathbb{N}$ , there exists  $u_I \in \tilde{S}_\mu$  such that

$$|u - u_I|_{H^1(\Omega)} \leq Ch_\mu^j \|u\|_{H^{j+1}(\Omega)}.$$

If  $u = 0$  on  $\partial\Omega$ , then we can take  $u_I \in S_\mu$ .

We now proceed to the proof of Theorem 0.1. We first need some lemmas. We begin with the following classical result [1, 16].

**Lemma 1.1.** *For  $v \in H^1(\Omega)$  there is a constant  $C$  that depends only on  $\Omega$  such that*

$$\|v\|_{H^1(\Omega)}^2 \leq C [ |v|_{H^1(\Omega)}^2 + \|v\|_{L^2(\partial\Omega)}^2 ].$$

From this lemma we obtain that  $|v_\mu|_{H^1(\Omega)}$  and  $\|v_\mu\|_{H^1(\Omega)}$  are equivalent norms on  $S_\mu$ , with equivalence bounds independent of  $\mu$ .

**Lemma 1.2.** *There exists  $C > 0$  such that*

$$C^{-1}|v_\mu|_{H^1(\Omega)} \leq \|v_\mu\|_{H^1(\Omega)} \leq C|v_\mu|_{H^1(\Omega)}$$

for all  $\mu$  large enough and all  $v_\mu \in S_\mu$ .

*Proof.* From Lemma 1.1, we have

$$\|v_\mu\|_{H^1(\Omega)}^2 \leq C[|v_\mu|_{H^1(\Omega)}^2 + \|v_\mu\|_{L^2(\partial\Omega)}^2] \leq C|v_\mu|_{H^1(\Omega)}^2 + Ch_\mu^{2m}\|v_\mu\|_{H^1(\Omega)}^2,$$

where the last inequality is a consequence of Assumption 1. Therefore, for  $\mu$  large,  $h_\mu$  is small enough and we get

$$\|v_\mu\|_{H^1(\Omega)} \leq C(1 - Ch_\mu^{2m})^{-1/2}|v_\mu|_{H^1(\Omega)}$$

which is enough to complete the proof.  $\square$

Lemma 1.2 allows us now to introduce the discrete solution  $u_\mu$  of Equation (1) using the standard procedure. Let  $B(v, w) := \int_\Omega \nabla v \cdot \nabla w dx$  be the usual bilinear form. For  $\mu$  large, let us define the discrete solution  $u_\mu \in S_\mu$  of the Poisson problem (1) by the usual formula

$$(3) \quad B(u_\mu, v_\mu) = \int_\Omega f(x)v_\mu(x)dx, \quad \text{for all } v_\mu \in S_\mu.$$

Let  $\nu$  be the outer unit normal to  $\partial\Omega$  and  $dS$  denote the surface measure on  $\partial\Omega$ . Similarly, let  $w_\mu \in S_\mu$ , for  $\mu$  large, be the solution of the variational problem

$$(4) \quad B(w_\mu, v_\mu) = \int_{\partial\Omega} \partial_\nu u(x)v_\mu(x)dS(x), \quad \text{for all } v_\mu \in S_\mu,$$

where  $u$  is the solution of Equation (1). Note that we need Lemma 1.2 to justify the existence and uniqueness of the (weak) solutions  $u_\mu$  and  $w_\mu$ .

**Lemma 1.3.** *Let  $u$  be the solution of the Poisson problem (1) and let  $u_\mu$  and  $w_\mu$  be as in Equations (3) and (4). Then  $B(u - u_\mu - w_\mu, v_\mu) = 0$  for all  $v_\mu \in S_\mu$ ; hence*

$$|u - u_\mu - w_\mu|_{H^1(\Omega)} \leq |u - v_\mu|_{H^1(\Omega)} \quad \text{for all } v_\mu \in S_\mu.$$

*Proof.* This is obtained from Assumption 1 as follows

$$(5) \quad B(u, v_\mu) = \int_\Omega \nabla u \cdot \nabla v_\mu dx = \int_\Omega f v_\mu dx + \int_{\partial\Omega} (\partial_\nu u) v_\mu dS(x) = B(u_\mu + w_\mu, v_\mu),$$

for all  $v_\mu \in S_\mu$ .  $\square$

We now proceed to estimate  $u_\mu$  and  $w_\mu$ .

**Lemma 1.4.** *Let  $u$  be the solution of the Poisson problem (1) and let  $w_\mu$  be the solution of the weak problem (4). Then, for  $\mu$  large, we have*

$$(6) \quad \|w_\mu\|_{H^1(\Omega)} \leq Ch_\mu^m \|u\|_{H^1(\Omega)},$$

with  $C$  a constant independent of  $\mu$  and  $u$ .

*Proof.* Let us first assume that  $f \in L^2(\Omega)$ . Then  $u \in H^2(\Omega)$  and hence  $\partial_\nu u$  is defined and belongs to  $H^{1/2}(\partial\Omega)$ . Green's formula gives  $\int_{\partial\Omega} \partial_\nu u(x) w(x) dS(x) = \int_\Omega (\Delta u w + \nabla u \cdot \nabla w) dx$ . Since the map  $H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$  is surjective with a continuous splitting, we see that

$$(7) \quad \|\partial_\nu u\|_{H^{-1/2}(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}.$$

By continuity, this inequality will continue to hold for  $f \in H^{-1}(\Omega)$ .

We have

$$\begin{aligned} \|w_\mu\|_{H^1(\Omega)}^2 &\leq C |w_\mu|_{H^1(\Omega)}^2 = CB(w_\mu, w_\mu) = C \int_{\partial\Omega} \partial_\nu u(x) w_\mu(x) dS(x) \\ &\leq C \|\partial_\nu u\|_{H^{-1/2}(\partial\Omega)} \|w_\mu\|_{H^{1/2}(\partial\Omega)} \leq Ch_\mu^m \|u\|_{H^1(\Omega)} \|w_\mu\|_{H^1(\Omega)}. \end{aligned}$$

Therefore  $\|w_\mu\|_{H^1(\Omega)} \leq Ch_\mu^m \|u\|_{H^1(\Omega)}$ , as claimed.  $\square$

From this lemma we obtain the following.

**Lemma 1.5.** *For  $\mu$  large, the solution  $u_\mu$  of the weak problem (3) satisfies*

$$(8) \quad \|u_\mu\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)},$$

with  $C$  a constant independent of  $\mu$  and  $u$ .

*Proof.* Let us first observe that Lemmas 1.2 and 1.3 and Equation (7) give

$$\begin{aligned} \|u_\mu\|_{H^1(\Omega)}^2 &\leq C |u_\mu|_{H^1(\Omega)}^2 = CB(u_\mu, u_\mu) = C[B(u, u_\mu) - B(w_\mu, u_\mu)] \\ &= C[B(u, u_\mu) - \langle \partial_\nu u, u_\mu \rangle_{\partial\Omega}] \leq C[|B(u, u_\mu)| + |\langle \partial_\nu u, u_\mu \rangle_{\partial\Omega}|] \\ &\leq C \|u\|_{H^1(\Omega)} \|u_\mu\|_{H^1(\Omega)} + C \|\partial_\nu u\|_{H^{-1/2}(\partial\Omega)} \|u_\mu\|_{H^{1/2}(\partial\Omega)} \\ &\leq C \|u\|_{H^1(\Omega)} \|u_\mu\|_{H^1(\Omega)} + Ch_\mu^m \|u\|_{H^1(\Omega)} \|u_\mu\|_{H^1(\Omega)}. \end{aligned}$$

Now it is easy to see that  $\|u_\mu\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)}$ , as claimed.  $\square$

We are ready now to prove Theorem 0.1.

*Proof.* We shall assume  $p = m$ , for simplicity. The proof in general is exactly the same. Lemma 1.3 and the projection property, together with Lemma 1.4, give

$$(9) \quad \begin{aligned} |u - u_\mu|_{H^1(\Omega)} &\leq |u - u_\mu - w_\mu|_{H^1(\Omega)} + |w_\mu|_{H^1(\Omega)} \\ &\leq |u - u_I|_{H^1(\Omega)} + Ch_\mu^m \|u\|_{H^1(\Omega)} \leq Ch_\mu^m \|u\|_{H^{m+1}(\Omega)}, \end{aligned}$$

where for the last line we also used the approximation property (Assumption 2).

The estimate in the  $H^1$ -norm is obtained from Lemma 1.1, Equation (9), Assumption 1, and Lemma 1.4 as follows

$$\begin{aligned} \|u - u_\mu\|_{H^1(\Omega)} &\leq C[|u - u_\mu|_{H^1(\Omega)} + \|u_\mu\|_{L^2(\partial\Omega)}] \\ &\leq Ch_\mu^m \|u\|_{H^{m+1}(\Omega)} + Ch_\mu^m \|u_\mu\|_{H^1(\Omega)} \leq Ch_\mu^m \|u\|_{H^{m+1}(\Omega)}. \end{aligned}$$

The proof is now complete.  $\square$

In view of some further applications, we now include an error estimate in a “negative order” Sobolev norm. We let  $H^{-l}(\Omega)$  to be the dual of  $H^l(\Omega)$  with pivot  $L^2(\Omega)$ . Since  $\Omega$  is a smooth domain,  $H^{-l}(\Omega)$  can also be described as the closure of  $\mathcal{C}^\infty(\Omega)$  in the norm

$$(10) \quad \|u\|_{H^{-l}(\Omega)} = \sup_{\phi \neq 0} \frac{|\langle u, \phi \rangle_{L^2(\Omega)}|}{\|\phi\|_{H^l(\Omega)}}$$

(Note that, in several other papers,  $H^{-l}(\Omega)$  denotes the dual of  $H_0^l(\Omega)$ .)

**Theorem 1.6.** *Let  $0 \leq l \leq m$ ,  $0 \leq p \leq m$ , and  $\gamma = \min\{l + p + 1, m\}$ . Then, under the assumptions of Theorem 0.1, the solutions  $u$  and  $u_\mu$  of Equation (1) and Equation (3), respectively, satisfy*

$$\|u - u_\mu\|_{H^{-l}(\Omega)} \leq Ch_\mu^\gamma \|u\|_{H^{p+1}(\Omega)} \leq Ch_\mu^\gamma \|f\|_{H^{p-1}(\Omega)},$$

for a constant  $C > 0$  independent of  $\mu$  and  $f \in H^{p-1}(\Omega)$ .

*Proof.* The proof of this theorem is an adaptation of the usual Nitsche-Aubin trick. Indeed, let us denote by  $F \in H^{l+2}(\Omega)$  the unique solution of the equation  $-\Delta F = \phi$ ,  $F = 0$  on  $\partial\Omega$ , for  $\phi \in H^l(\Omega)$  arbitrary, non-zero. Then there exists a constant  $C > 0$ , independent of  $\phi$ , such that  $\|F\|_{H^{l+2}(\Omega)} \leq C\|\phi\|_{H^l(\Omega)}$ . By Assumption 2, there exists  $F_I \in S_\mu$  such that

$$(11) \quad |F - F_I|_{H^1(\Omega)} \leq Ch_\mu^{l+1} \|F\|_{H^{l+2}(\Omega)}.$$

Then, the inequality (11) leads to the following easy observation, which will be used later,

$$(12) \quad |F_I|_{H^1(\Omega)} = |F - (F - F_I)|_{H^1(\Omega)} \leq |F|_{H^1(\Omega)} + Ch_\mu^{l+1} \|F\|_{H^{l+2}(\Omega)} \leq C\|\phi\|_{H^l(\Omega)}.$$

In the following calculation, we shall use Equation (5) in the first inequality, and then Theorem 0.1, Equations (11) and (12), 1.4, and 1.5 for the last inequality, to obtain

$$\begin{aligned} \|u - u_\mu\|_{H^{-l}(\Omega)} &= \sup_{\phi \neq 0} \frac{|(u - u_\mu, \phi)_{L^2(\Omega)}|}{\|\phi\|_{H^l(\Omega)}} = \sup_{\phi \neq 0} \frac{|B(u - u_\mu, F) + \int_{\partial\Omega} u_\mu \partial_\nu F dS|}{\|\phi\|_{H^l(\Omega)}} \\ &\leq \sup_{\phi \neq 0} \frac{|B(u - u_\mu, F - F_I)|}{\|\phi\|_{H^l(\Omega)}} + \sup_{\phi \neq 0} \frac{|B(w_\mu, F_I)|}{\|\phi\|_{H^l(\Omega)}} + \sup_{\phi \neq 0} \frac{|\int_{\partial\Omega} u_\mu \partial_\nu F dS|}{\|\phi\|_{H^l(\Omega)}} \\ &\leq \sup_{\phi \neq 0} \frac{|u - u_\mu|_{H^1(\Omega)} |F - F_I|_{H^1(\Omega)}}{\|\phi\|_{H^l(\Omega)}} + \sup_{\phi \neq 0} \frac{|w_\mu|_{H^1(\Omega)} |F_I|_{H^1(\Omega)}}{\|\phi\|_{H^l(\Omega)}} \\ &\quad + \sup_{\phi \neq 0} \frac{\|u_\mu\|_{L^2(\partial\Omega)} \|\partial_\nu F\|_{L^2(\partial\Omega)}}{\|\phi\|_{H^l(\Omega)}} \\ &\leq Ch_\mu^{p+l+1} \|u\|_{H^{p+1}(\Omega)} + Ch_\mu^m \|u\|_{H^1(\Omega)} + Ch_\mu^m \|u\|_{H^1(\Omega)} \\ &\leq Ch_\mu^\gamma \|u\|_{H^{p+1}(\Omega)}, \end{aligned}$$

by the definition of  $\gamma$ . This completes the proof.  $\square$

For  $p = 0$ , the proof of the above result requires the full strength of Assumption 1. The case  $p = 0$  is the one needed for the results of Section 6.

## 2. NON-HOMOGENEOUS BOUNDARY CONDITIONS

In this subsection we provide an approach to the non-homogeneous Dirichlet boundary conditions. That is, we consider the boundary value problem

$$(13) \quad \begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Our approach is to reduce it to the case  $g = 0$  and then to use the results on the Poisson problem (1). In a purely theoretical framework, this is achieved using an extension  $G$  of  $g$  and then solving the problem  $-\Delta w = f + \Delta G$ ,  $w = 0$  on  $\partial\Omega$ .

The solution of (13) will then be  $u = w + G$ . This gives that the problem (13) has a unique solution  $u \in H^{p+1}(\Omega)$  for any  $f \in H^{p-1}(\Omega)$  and  $g \in H^{1/2+p}(\partial\Omega)$  and it satisfies

$$\|u\|_{H^{p+1}(\Omega)} \leq C(\|f\|_{H^{p-1}(\Omega)} + \|g\|_{H^{1/2+p}(\partial\Omega)}),$$

for a constant  $C > 0$  that depends only on  $\Omega$  and  $p \in \mathbb{Z}_+$ . (This result is valid also for  $p = 0$ .)

In practice, however, we need to slightly modify this approach since it is not practical to construct the extension  $G$  (this is especially a problem if  $g$  has low regularity, that is, if  $g$  is a distribution, for instance). We will be looking therefore for a sequence  $G_k$  of *approximate extensions* of  $g$ , that is, satisfying the following assumption. Recall that the sequence  $h_\mu$  should be thought of as the “typical size” of the elements defining  $S_\mu$  and satisfies  $h_\mu \rightarrow 0$ .

**Assumption 3 (approximate extensions).** We assume that there exist a sequence of spaces  $\tilde{S}_k$ ,  $S_k \subset \tilde{S}_k$ , satisfying Assumption 2 and a constant  $C > 0$  such that, for any  $g \in H^{m+1/2}(\partial\Omega)$ , there exists a sequence  $G_k \in \tilde{S}_k$  such that  $\|G_k|_{\partial\Omega} - g\|_{H^{1/2}(\partial\Omega)} \leq Ch_k^m \|g\|_{H^{m+1/2}(\partial\Omega)}$  and  $\|G_k\|_{H^{m+1}(\Omega)} \leq C\|g\|_{H^{m+1/2}(\partial\Omega)}$ .

The proof of Theorem 0.2 follows below.

*Proof.* Remember that  $w_k$  was introduced as the exact solution to the boundary value problem  $-\Delta w_k = f + \Delta G_k$  in  $\Omega$ ,  $w_k = 0$  on  $\partial\Omega$ . Let  $(w_k)_\mu \in S_\mu$  be the approximate solution of this equation, as in Equation (2).

We have that  $v_k := w_k + G_k$  solves the boundary value problem

$$-\Delta v_k = f \text{ on } \Omega, \quad v_k = G_k \text{ on } \partial\Omega.$$

Hence the difference  $u - v_k$  solve the boundary value problem  $\Delta(u - v_k) = 0$ ,  $(u - v_k) = g - G_k$  on  $\partial\Omega$ . From this and Assumption 3 we obtain

$$(14) \quad \|u - v_k\|_{H^1(\Omega)} \leq C\|g - G_k\|_{H^{1/2}(\partial\Omega)} \leq Ch_k^m \|g\|_{H^{m+1/2}(\partial\Omega)}.$$

Theorem 0.1 and Assumption 3 then give

$$\begin{aligned} \|v_k - u_k\|_{H^1(\Omega)} &= \|w_k - (w_k)_k\|_{H^1(\Omega)} \leq Ch_k^m \|f + \Delta G_k\|_{H^{m-1}(\Omega)} \\ &\leq Ch_k^m (\|f\|_{H^{m-1}(\Omega)} + \|G_k\|_{H^{m+1}(\Omega)}) \leq Ch_k^m (\|f\|_{H^{m-1}(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)}). \end{aligned}$$

Hence

$$(15) \quad \|v_k - u_k\|_{H^1(\Omega)} = \|w_k - (w_k)_k\|_{H^1(\Omega)} \leq Ch_k^m (\|f\|_{H^{m-1}(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)}).$$

From Equations (14) and (15) we obtain  $\|u - u_k\|_{H^1(\Omega)} \leq Ch_k^m (\|f\|_{H^{m-1}(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)})$ , which is what we had to prove.  $\square$

### 3. DISTRIBUTIONAL BOUNDARY DATA AND THE “INF-SUP” CONDITION

Let us consider the Dirichlet problem (13) (*i.e.*,  $-\Delta u = f$  in  $\Omega$  and  $u = g$  on  $\partial\Omega$ ), with  $g \in H^{1/2-s}(\partial\Omega)$  and  $f \in H^{-1-s}(\Omega)$ ,  $s \in \mathbb{R}$ . If  $s \leq 0$ , it is well known that the boundary value problem (13) has a unique solution  $u \in H^{1-s}(\Omega)$ . Moreover, there is a constant  $C_{\Omega,s}$ , depending only on  $\Omega$  and  $s \leq 0$ , such that

$$\|u\|_{H^{1-s}(\Omega)} \leq C_{\Omega,s} (\|f\|_{H^{-1-s}(\Omega)} + \|g\|_{H^{1/2-s}(\partial\Omega)}).$$

In this section, we extend the above result to the case when  $g \in H^{1/2-s}(\partial\Omega)$ , with  $s > 0$ . Our approach is based on the so called “inf-sup” condition [3].

**3.1. The weak formulation.** Let us define the functional space  $\tilde{H}^{1-s}(\Omega) := H^{1-s}(\Omega) \oplus H^{-1/2-s}(\partial\Omega)$ . Intuitively, for an element  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$ , the first component  $u_0$  should be thought of as  $u$  in the interior of  $\Omega$ , while the second component  $u_1$  represents the normal derivative  $\partial_\nu u$  on  $\partial\Omega$ . Let  $\tilde{B} : \tilde{H}^{1-s}(\Omega) \times H^{1+s}(\Omega) \rightarrow \mathbb{C}$  be the bilinear functional defined by

$$\tilde{B}(u, v) := \langle u_0, \Delta v \rangle_\Omega + \langle u_1, v \rangle_{\partial\Omega},$$

where  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$ .

**Definition 3.1.** Let  $g \in H^{1/2-s}(\partial\Omega)$  and  $f \in H^{-1-s}(\Omega)$ ,  $s \in \mathbb{R}$ . We say that  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$  satisfies (13) in *weak sense*, or that  $u$  is a *weak solution* of the Dirichlet problem (13), if

$$(16) \quad \tilde{B}(u, v) = -\langle f, v \rangle_\Omega + \langle g, \partial_\nu v \rangle_{\partial\Omega},$$

for all  $v \in H^{1+s}(\Omega)$ .

*Remark 3.2.* If  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$  is a weak solution of the Dirichlet problem (13) in the sense of the above definition for  $s \leq 0$ , then it is easy to see that the first component  $u_0$  is a classical solution for (13) and  $u_1 = \partial_\nu u_0$  on  $\partial\Omega$ .

*Remark 3.3.* If  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$  is a solution of (13) in weak sense, then it is unique with this property.

The main ingredient we will use for proving Theorem 0.3 is the following “inf-sup” lemma (or the Babuška–Brezzi condition) [2, 3]. (This result was used for similar purposes in [9] in order to deal with low regularity Neumann data.)

**Theorem 3.4.** Let  $X$  and  $Y$  be reflexive Banach spaces with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ . Also, let  $B_1 : X \times Y \rightarrow \mathbb{C}$  be a bilinear form. Assume that

- (a)  $B_1$  is continuous;
- (b) There exists  $\gamma > 0$  such that  $\inf_{\|x\|=1} \sup_{\|y\|\leq 1} |B_1(x, y)| \geq \gamma$ ;
- (c)  $\sup_{\|x\|_X \leq 1} |B_1(x, y)| > 0$  whenever  $y \neq 0$ .

Then for any continuous functional  $F : Y \rightarrow \mathbb{C}$  there exists a unique  $x \in X$  such that  $F(y) = B_1(x, y)$ , for all  $y \in Y$ . Moreover, we have  $\|x\| \leq \|F\|/\gamma$ .

We are ready now to prove the main result of this section, that is, Theorem 0.3.

*Proof.* It is easy to see that  $\tilde{B}$  is continuous from its definition and the definition of negative order Sobolev spaces. Therefore, Condition (a) in Theorem 3.4 is satisfied. Let  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$  be such that  $\|u\| := \|u_0\|_{H^{1-s}(\Omega)} + \|u_1\|_{H^{-1/2-s}(\partial\Omega)} = 1$ . Since  $u$  also belongs to  $(H^{-1+s}(\Omega) \oplus H^{1/2+s}(\partial\Omega))^*$ , which is the dual of  $\tilde{H}^{1-s}(\Omega)$ , there exists  $(v, v_1) \in H^{-1+s}(\Omega) \oplus H^{1/2+s}(\partial\Omega)$  with  $\|(v, v_1)\| := \|v\|_{H^{-1+s}(\Omega)} + \|v_1\|_{H^{1/2+s}(\partial\Omega)} = 1$  such that

$$(17) \quad \langle u, (v, v_1) \rangle := \langle u_0, v \rangle_\Omega + \langle u_1, v_1 \rangle_{\partial\Omega} \geq 1/2.$$

Let  $V \in H^{1+s}(\Omega) \cap H^{1/2+s}(\partial\Omega)$  be the unique solution of the inhomogeneous Dirichlet problem

$$(18) \quad \begin{cases} \Delta V = v & \text{in } \Omega, \\ V = v_1 & \text{on } \partial\Omega. \end{cases}$$

Then,

$$(19) \quad \tilde{B}(u, V) = \langle u_0, \Delta V \rangle_\Omega + \langle u_1, V \rangle_{\partial\Omega} = \langle u_0, v \rangle_\Omega + \langle u_1, v_1 \rangle_{\partial\Omega} \geq 1/2,$$

and this inequality implies that Condition (b) in Theorem 3.4 is also satisfied.

Finally, let us check Condition (c) in Theorem 3.4. Let  $v \in H^{1+s}(\Omega)$  such that  $\tilde{B}(u, v) = 0$ , for all  $u = (u_0, u_1) \in \tilde{H}^{1-s}(\Omega)$ . Then,  $v$  must satisfy the Dirichlet problem

$$(20) \quad \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

whose unique solution is  $v = 0$ . This shows that Condition (c) in Theorem 3.4 is satisfied as well. The conclusion of the Theorem 0.3 follows if we take the functional  $F$  in Theorem 3.4 to be  $F(v) := -\langle f, v \rangle_\Omega + \langle g, \partial_\nu v \rangle_{\partial\Omega}$ .  $\square$

## Part 2. GFEM Approximation Spaces

### 4. THE GENERALIZED FINITE ELEMENT METHOD

Our goal is to construct a sequence  $S_\mu$ ,  $\mu = 1, 2, \dots$ , of Generalized Finite Element spaces that satisfy the two assumptions of the previous section. To this end, we shall introduce a sequence of Generalized Finite Element spaces that satisfy certain conditions (Conditions A( $h_\mu$ ), B, C, and D). In the following sections we shall prove that these conditions imply Assumptions 1 and 2.

We begin by recalling a few basic facts about the Generalized Finite Element Method [4, 8, 26]. This method is quite convenient when one needs test or trial spaces with high regularity.

**4.1. Basic facts.** Let  $k \in \mathbb{Z}_+$ . We shall denote as usual

$$|u|_{W^{k,\infty}(\Omega)} := \max_{|\alpha|=k} \|\partial^\alpha u\|_{L^\infty(\Omega)}, \quad \|u\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)},$$

$W^{k,\infty}(\Omega) := \{u, \|u\|_{W^{k,\infty}(\Omega)} < \infty\}$ , and  $\|\nabla \omega\|_{W^{k,\infty}(\Omega)} := \sum_j \|\partial_j \omega\|_{W^{k,\infty}(\Omega)}$ . In particular,  $|u|_{W^{0,\infty}(\Omega)} = \|u\|_{W^{0,\infty}(\Omega)} = \|u\|_{L^\infty(\Omega)}$ .

We shall need the following slight generalization of a definition from [8, 26]:

**Definition 4.1.** Let  $\Omega \subset \mathbb{R}^n$  be an open set and  $\{\omega_j\}_{j=1}^N$  be an open cover of  $\Omega$  such that any  $x \in \Omega$  belongs to at most  $\kappa$  of the sets  $\omega_j$ . Also, let  $\{\phi_j\}$  be a partition of unity consisting of  $W^{m,\infty}(\Omega)$  functions and subordinated to the covering  $\{\omega_j\}$  (i.e.,  $\text{supp } \phi_j \subset \bar{\omega}_j$ ). If

$$(21) \quad \|\partial^\alpha \phi_j\|_{L^\infty(\Omega)} \leq C_k / (\text{diam } \omega_j)^k, \quad k = |\alpha| \leq m,$$

for any  $j = 1, \dots, N$ , then  $\{\phi_j\}$  is called a  $(\kappa, C_0, C_1, \dots, C_m)$  *partition of unity*.

Assume also that we are given linear subspaces  $\Psi_j \subset H^m(\omega_j)$ ,  $j = 1, 2, \dots, N$ . The spaces  $\Psi_j$  will be called *local approximation spaces* and will be used to define the space

$$(22) \quad S = S_{GFEM} := \left\{ \sum_{j=1}^N \phi_j v_j, v_j \in \Psi_j \right\} \subset H^m(\Omega),$$

which will be called the *GFEM-space*. The set  $\{\omega_j, \phi_j, \Psi_j\}$  will be called the *set of data defining the GFEM-space*  $S$ . A basic approximation property of the GFEM-spaces is the following Theorem from [8].

**Theorem 4.2** (Babuška-Melenk). *We shall use the notations and definitions of Definition 4.1 and after. Let  $\{\phi_j\}$  be a  $(\kappa, C_0, C_1)$  partition of unity. Also, let  $v_j \in \Psi_j \subset H^1(\omega_j)$ ,  $u_{ap} := \sum_j \phi_j v_j \in S$ , and  $d_j = \text{diam } \omega_j$ , the diameter of  $\omega_j$ . Then*

$$(23) \quad \begin{aligned} \|u - u_{ap}\|_{L^2(\Omega)}^2 &\leq \kappa C_0^2 \sum_j \|u - v_j\|_{L^2(\omega_j)}^2 \quad \text{and} \\ \|\nabla(u - u_{ap})\|_{L^2(\Omega)}^2 &\leq 2\kappa \sum_j \left( \frac{C_1^2 \|u - v_j\|_{L^2(\omega_j)}^2}{(d_j)^2} + C_0^2 \|\nabla(u - v_j)\|_{L^2(\omega_j)}^2 \right). \end{aligned}$$

**4.2. Conditions on GFEM data defining  $S_\mu$ .** Recall that  $\omega$  is *star-shaped* with respect to  $\omega^* \subset \omega$  if, for every  $x \in \omega$  and every  $y \in \omega^*$ , the segment with end points  $x$  and  $y$  is completely contained in  $\omega$ . Also, recall that we have fixed an integer  $m$  that plays the role of the order of approximation. Let  $\{\omega_j, \phi_j, \Psi_j\}_{j=1}^N$  be a *single*, fixed data defining a GFEM-space  $S$ , as in the previous subsection, and let  $\Sigma := \{\omega_j, \phi_j, \Psi_j, \omega_j^*\}$ , where  $\omega_j$  is star-shaped with respect to  $\omega_j^* \subset \omega_j$ . We shall need, in fact, to consider a sequence of such data

$$(24) \quad \Sigma_\mu = \{\omega_j^\mu, \phi_j^\mu, \Psi_j^\mu, \omega_j^{*\mu}\}_{j=1}^{N_\mu}, \quad \mu \in \mathbb{N},$$

defining GFEM-spaces  $S_\mu$

$$(25) \quad S_\mu := \left\{ \sum_{j=1}^{N_\mu} \phi_j^\mu v_j, v_j \in \Psi_j^\mu \right\} \subset H^m(\Omega),$$

such that there exist constants  $A, C_j, \sigma$ , and  $\kappa$  and a sequence  $h_\mu \rightarrow 0$ , as  $\mu \rightarrow \infty$ , for which  $\Sigma_\mu$  satisfies Conditions A( $h_\mu$ ), B, C, and D below for  $\mu \in \mathbb{N}$ . The sequence  $h_\mu$  gives the “typical size” of the elements defining  $S_\mu$ , as in the first part.

**Condition A( $h_\mu$ ).** *We have that  $\Omega = \bigcup_{j=1}^{N_\mu} \omega_j^\mu$  and for each  $j = 1, 2, \dots, N_\mu$ , the set  $\omega_j^\mu$  is open of diameter  $d_j^\mu \leq h_\mu \leq 1$  and  $\omega_j^{*\mu} \subset \omega_j^\mu$  is an open ball of diameter  $\geq \sigma d_j^\mu$  such that  $\omega_j^\mu$  is star-shaped with respect to  $\omega_j^{*\mu}$ .*

Notice that we only assume the open covering  $\{\omega_j^\mu\}$  to be *nondegenerate*, a weaker condition than *quasi-uniformity* (see [16], Section 4.4, for definitions and more information on these notions).

**Condition B.** *The family  $\{\phi_j^\mu\}_{j=1}^{N_\mu}$  is a  $(\kappa, C_0, C_1, \dots, C_m)$  partition of unity.*

The following condition defines the local approximation spaces  $\Psi_j^\mu$ . To formulate this condition, let us choose  $x_j \in \overline{\omega_j^\mu} \cap \partial\Omega$ , if the intersection is not empty. We can assume that linear coordinates have been chosen such that  $x_j = 0$  and the tangent space to  $\partial\Omega$  at  $x_j$  is  $\{x_n = 0\} = \mathbb{R}^{n-1}$ . For  $h_\mu$  small, we can assume that  $\overline{\omega_j^\mu} \cap \partial\Omega$  is contained in the graph of a smooth function  $g_j^\mu : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ . If  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , then we shall denote  $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ , so that  $x = (x', x_n)$ . Let  $q_j^\mu : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  be a polynomial of order  $m$  such that

$$(26) \quad \begin{aligned} |g_j^\mu(x') - q_j^\mu(x')| &\leq C(d_j^\mu)^{m+1} \quad \text{and} \\ |\nabla g_j^\mu(x') - \nabla q_j^\mu(x')| &\leq C(d_j^\mu)^m \quad \text{for all } (x', x_n) \in \omega_j^\mu. \end{aligned}$$

This condition is satisfied, for instance, if  $\partial^\alpha g_j^\mu(0) = \partial^\alpha q_j^\mu(0)$ , for all  $|\alpha| \leq m$ . In this case, the  $m$ -degree polynomial  $q_j^\mu : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  is uniquely defined by the aforementioned requirement.

Next, denote by  $\tilde{q}_j^\mu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  the bijective map

$$(27) \quad \tilde{q}_j^\mu(x) = \tilde{q}_j^\mu(x', x_n) = (x', x_n + q_j^\mu(x')).$$

Let us denote by  $\mathcal{P}_k$  the space of polynomials of order at most  $k$  in  $n$  variables.

**Condition C.** We have  $\Psi_j^\mu = \mathcal{P}_m$  if  $\overline{\omega_j^\mu} \cap \partial\Omega = \emptyset$  and, otherwise,

$$\Psi_j^\mu = \{p \circ (\tilde{q}_j^\mu)^{-1}, p \in \mathcal{P}_m, \text{ such that } p(x', 0) = 0\},$$

where  $q_j^\mu$  are polynomials satisfying Equation (26) with a constant  $C$  independent of  $j$  and  $\mu$ .

An equivalent form of the condition “ $p \in \mathcal{P}_m, p(x', 0) = 0$ ” is “ $p = x_n p_1, p_1 \in \mathcal{P}_{m-1}$ ,” because any polynomial vanishing on the hyperplane  $\{x_n = 0\}$  is a multiple of  $x_n$ . Since  $(\tilde{q}_j^\mu)^{-1}(x', x_n) = (x', x_n - q_j^\mu(x'))$ , we obtain  $p(x', x_n) = (x_n - q_j^\mu(x'))p_1 \circ (\tilde{q}_j^\mu)^{-1}$ .

**Condition D.** We have  $\phi_j^\mu = 1$  on  $\omega_j^{*\mu}$  for all  $j = 1, \dots, N_\mu$  for which  $\overline{\omega_j^\mu} \cap \partial\Omega \neq \emptyset$ .

The constants  $C_j$ ,  $\sigma$ , and  $\kappa$  will be called *structural constants*. Note that we must have  $N_\mu \rightarrow \infty$  as  $\mu \rightarrow \infty$ .

The above assumptions are slightly weaker than the ones introduced in [9]. For instance, Condition C implies the following property (which is similar to Condition C in [9])

For any  $w \in \Psi_j^\mu$ , any  $0 \leq l \leq m+1$ , and any ball  $\omega^* \subset \omega_j^\mu$  of diameter  $\geq \sigma d_j^\mu$ .

$$(28) \quad \|w\|_{H^l(\omega_j^\mu)} \leq C \|w\|_{H^l(\omega^*)}.$$

For further applications, we shall also need a variant of the spaces  $S_\mu$  in which *no boundary conditions are imposed*. Recall the functions  $q_j^\mu$  used to define the spaces  $\Psi_j^\mu$ . Let  $\tilde{\Psi}_j^\mu = \Psi_j^\mu$  if  $\omega_j$  does not touch the boundary  $\partial\Omega$  and  $\tilde{\Psi}_j^\mu = \{p \circ (\tilde{q}_j^\mu)^{-1}, p \in \mathcal{P}_m\}$  otherwise (the difference is that we no longer require  $p$  to vanish when  $x_n = 0$ ). We then define

$$(29) \quad \tilde{S}_\mu := \left\{ \sum_{j=1}^{N_\mu} \phi_j^\mu v_j, v_j \in \tilde{\Psi}_j^\mu \right\} \subset H^m(\Omega).$$

We shall also need the following standard lemma, a proof of which, for  $s \in \mathbb{Z}_+$ , can be found in [9]. For  $s \geq 0$  it is proved by interpolation.

**Lemma 4.3.** Let  $\psi_j$  be measurable functions defined on an open set  $W$  and  $s \geq 0$ . Assume that there exists an integer  $\kappa$  such that a point  $x \in W$  can belong to no more than  $\kappa$  of the sets  $\text{supp}(\psi_j)$ . Let  $f = \sum_j \psi_j$ . Then there exists a constant  $C > 0$ , depending only on  $\kappa$ , such that  $\|f\|_{H^s(W)}^2 \leq C \sum_j \|\psi_j\|_{H^s(W)}^2$ .

Recall that  $d_j^\mu$  denotes the diameter of  $\omega_j^\mu$ . Let us observe that Condition A( $h_\mu$ ) implies the following inverse inequality.

**Lemma 4.4.** *There exists  $C > 0$ , depending only on  $\sigma$ , such that*

$$(30) \quad \|p\|_{H^s(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s} \|p\|_{H^r(\omega_j^\mu)},$$

for all  $0 \leq r \leq s \leq m$ , all  $j$ , all  $\mu$ , and all polynomials  $p$  of order  $m$ .

*Proof.* The proof of this lemma is inspired from the proof of (4.5.3) Lemma of [16].

Consider  $\mu$  and  $1 \leq j \leq N_\mu$  arbitrary, but fixed for the moment. Let

$$\hat{\omega}_j^\mu := \left\{ \frac{1}{d_j^\mu}(x - x_j^\mu), x \in \omega_j^\mu \right\}, \quad \hat{\omega}_j^{*\mu} := \left\{ \frac{1}{d_j^\mu}(x - x_j^\mu), x \in \omega_j^{*\mu} \right\},$$

where  $x_j^\mu$  is the center of the ball  $\omega_j^{*\mu}$ .

If  $p \in \mathcal{P}_m$  is a polynomial of order  $m$ , then  $\hat{p}$  is defined by  $\hat{p}(\hat{x}) := p(d_j^\mu \hat{x} + x_j^\mu)$  for all  $\hat{x}$ . Observe that the set  $\hat{\mathcal{P}}_m := \{\hat{p} : p \in \mathcal{P}_m\}$  is nothing but the set of all  $m$ -degree polynomials in  $\hat{x}$ . Clearly,

$$(31) \quad |\hat{p}|_{H^k(\hat{\omega}_j^\mu)} = (d_j^\mu)^{k-n/2} |p|_{H^k(\omega_j^\mu)}, \quad \text{for } 0 \leq k \leq m.$$

We first prove (30) for the case  $r = 0$ . Since  $\hat{\mathcal{P}}_m$  is finite dimensional, we have by the equivalence of norms on the unit ball  $B(0, 1)$  that

$$(32) \quad \|\hat{p}\|_{H^k(B(0,1))} \leq C \|\hat{p}\|_{L^2(B(0,1))}, \quad \text{for any } 0 \leq k \leq m,$$

where  $C > 0$  is a constant that does not depend on  $k$ ,  $j$ , and  $\mu$ . From Condition A( $h_\mu$ ), we obtain that

$$(33) \quad \|\hat{p}\|_{L^2(B(0,1))} \leq C \|\hat{p}\|_{L^2(\hat{\omega}_j^{*\mu})}$$

where  $C > 0$  depends only on the structural constant  $\sigma$ . From (32) and (33), it is clear that

$$\|\hat{p}\|_{H^k(\hat{\omega}_j^\mu)} \leq C \|\hat{p}\|_{L^2(\hat{\omega}_j^\mu)} \quad \forall \hat{p} \in \hat{\mathcal{P}}_m,$$

where  $C > 0$  depends only on  $\sigma$ . Therefore, (31) implies

$$|p|_{H^k(\omega_j^\mu)} (d_j^\mu)^{k-n/2} \leq C \|p\|_{L^2(\omega_j^\mu)} (d_j^\mu)^{-n/2} \quad \text{for } 0 \leq k \leq s,$$

from which we deduce that

$$|p|_{H^k(\omega_j^\mu)} \leq C (d_j^\mu)^{-k} \|p\|_{L^2(\omega_j^\mu)} \quad \text{for } 0 \leq k \leq s.$$

Since  $d_j^\mu \leq h_\mu \leq 1$ , we have

$$(34) \quad \|p\|_{H^s(\omega_j^\mu)} \leq C (d_j^\mu)^{-s} \|p\|_{L^2(\omega_j^\mu)},$$

which is just (30) for  $r = 0$ .

Let us now analyse the general case  $0 \leq r \leq s \leq m$ . For  $|\alpha| = k$ , with  $s - r \leq k \leq s$ ,  $D^\alpha p = D^\beta D^\gamma p$  for  $|\beta| = s - r$  and  $|\gamma| = k + r - s$ . Therefore,

$$\begin{aligned} \|D^\alpha p\|_{L^2(\omega_j^\mu)} &\leq \|D^\gamma p\|_{H^{s-r}(\omega_j^\mu)} \\ &\leq C (d_j^\mu)^{r-s} \|D^\gamma p\|_{L^2(\omega_j^\mu)} \quad (\text{by (34)}) \\ &\leq C (d_j^\mu)^{r-s} |p|_{H^{k+r-s}(\omega_j^\mu)}. \end{aligned}$$

Since

$$|p|_{H^k(\omega_j^\mu)} := \sum_{|\alpha|=k} \|D^\alpha p\|_{L^2(\omega_j^\mu)},$$

we obtain that

$$|p|_{H^k(\omega_j^\mu)} \leq C (d_j^\mu)^{r-s} |p|_{H^{k+r-s}(\omega_j^\mu)} \quad \text{for } s - r \leq k \leq s.$$

This implies that

$$(35) \quad |p|_{H^k(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s} \|p\|_{H^r(\omega_j^\mu)} \quad \text{for } s - r \leq k \leq s.$$

From (34), we also have

$$(36) \quad \|p\|_{H^{s-r}(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s} \|p\|_{L^2(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s} \|p\|_{H^r(\omega_j^\mu)}.$$

Combining (35) and (36) gives (30) and this ends the proof of the lemma.  $\square$

## 5. PROPERTIES OF THE SPACES $S_\mu$

In this section, we establish some properties of the GFEM spaces  $S_\mu$ ,  $\mu \in \mathbb{N}$ , defined in Equation (25) using the data  $\Sigma_\mu = \{\omega_j^\mu, \phi_j^\mu, \Psi_j^\mu, \omega_j^{*\mu}\}_{j=1}^{N_\mu}$  satisfying conditions A( $h_\mu$ ), B, C, and D introduced in the previous section for  $h_\mu \rightarrow 0$ . The main result is that the sequence  $S_\mu$  satisfies Assumptions 1 and 2 of the first section. Also, we prove that there is a sequence  $G_k \in \tilde{S}_k$  of approximate extensions of  $g$  which satisfies Assumption 3 in the case of the non-homogeneous Dirichlet boundary-value problem (13).

Hereafter, for simplicity, we will omit the index  $\mu$  whenever its appearance is implicit.

Let us fix  $j$  such that  $\bar{\omega}_j \cap \partial\Omega$  is not empty. Recall the functions  $g_j, q_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  defined in the previous section. So, for  $h$  small,  $\bar{\omega}_j \cap \partial\Omega$  is contained in  $\{(x', g_j(x'))\}$ , the graph of the smooth function  $g_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  (this may require a preliminary rotation, which is not included in the notation, however, for the sake of simplicity). Let  $\tilde{q}_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the bijective map defined by Equation (27). Similarly, let

$$(37) \quad \tilde{g}_j(x) = \tilde{g}_j(x', x_n) = (x', x_n + g_j(x')).$$

Then  $\tilde{g}_j$  maps  $\mathbb{R}^{n-1}$  to a surface containing  $\bar{\omega}_j \cap \partial\Omega$ . We have  $\tilde{g}_j^{-1}(x) = (x', x_n - g_j(x'))$  and  $\tilde{q}_j^{-1}(x) = (x', x_n - q_j(x'))$ .

We shall need the following estimate.

**Lemma 5.1.** *For any polynomial  $p$  of order  $m$ , we have*

$$\begin{aligned} \|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} &\leq C d_j^{m+1} \|p\|_{H^1(\omega_j)} \quad \text{and} \\ \|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} &\leq C d_j^m \|p\|_{H^1(\omega_j)}, \end{aligned}$$

where  $C$  is a constant independent of  $p$ ,  $\mu$ , and  $j$ .

*Proof.* By Taylor's expansion theorem in the  $x_n$  variable, we have

$$\begin{aligned} p \circ \tilde{g}_j^{-1}(x', x_n) &= p(x', x_n - g_j(x')) = p(x', x_n) - g_j(x') \partial_n p(x', x_n) + \dots \\ &\quad + (-1)^k \frac{g_j(x')^k}{k!} \partial_n^k p(x', x_n) + \dots + (-1)^m \frac{g_j(x')^m}{m!} \partial_n^m p(x', x_n) \end{aligned}$$

and

$$\begin{aligned} p \circ \tilde{q}_j^{-1}(x', x_n) &= p(x', x_n - q_j(x')) = p(x', x_n) - q_j(x') \partial_n p(x', x_n) + \dots \\ &\quad + (-1)^k \frac{q_j(x')^k}{k!} \partial_n^k p(x', x_n) + \dots + (-1)^m \frac{q_j(x')^m}{m!} \partial_n^m p(x', x_n). \end{aligned}$$

Then,

$$\begin{aligned} |p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)| &= |p(x', x_n - g_j(x')) - p(x', x_n - q_j(x'))| \\ &\leq |g_j(x') - q_j(x')| \cdot |\partial_n p(x', x_n)| + \dots + \left| \frac{g_j(x')^k - q_j(x')^k}{k!} \right| \cdot |\partial_n^k p(x', x_n)| + \dots \\ &\quad + \left| \frac{g_j(x')^m - q_j(x')^m}{m!} \right| \cdot |\partial_n^m p(x', x_n)|. \end{aligned}$$

From this and the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned} |p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)|^2 &= |p(x', x_n - g_j(x')) - p(x', x_n - q_j(x'))|^2 \\ &\leq m[(g_j(x') - q_j(x'))^2 \partial_n p(x', x_n)^2 + \dots + \frac{(g_j(x')^k - q_j(x')^k)^2}{(k!)^2} \partial_n^k p(x', x_n)^2 + \dots \\ &\quad + \frac{(g_j(x')^m - q_j(x')^m)^2}{(m!)^2} \partial_n^m p(x', x_n)^2]. \end{aligned}$$

Notice that  $|g_j(x')| \leq d_j$ , for all  $(x', x_n) \in \omega_j$ , and because  $q_j(x') = g_j(x') + O(d_j^{m+1})$ , for all  $(x', x_n) \in \omega_j$ , we have

$$(g_j(x')^k - q_j(x')^k)^2 = [g_j^k(x') - (g_j(x') + O(d_j^{m+1}))^k]^2 \leq C d_j^{2(m+k)}, \text{ for } k = 1, \dots, m,$$

which in turn implies that

$$\begin{aligned} |p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)|^2 &\leq C d_j^{2(m+1)} [\partial_n p(x', x_n)^2 + d_j^2 \partial_n^2 p(x', x_n)^2 + \dots \\ &\quad + d_j^{2(k-1)} \partial_n^k p(x', x_n)^2 + \dots + d_j^{2(m-1)} \partial_n^m p(x', x_n)^2]. \end{aligned}$$

By using the inverse inequality  $d_j^{k-1} \|p\|_{H^k(\omega_j)} \leq C \|p\|_{H^1(\omega_j)}$ , we get

$$\begin{aligned} \|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)}^2 &= \int_{\omega_j} |p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)|^2 dx' dx_n \\ &\leq C d_j^{2(m+1)} \int_{\omega_j} [\partial_n p(x', x_n)^2 + d_j^2 \partial_n^2 p(x', x_n)^2 + \dots + d_j^{2(k-1)} \partial_n^k p(x', x_n)^2 + \dots \\ &\quad + d_j^{2(m-1)} \partial_n^m p(x', x_n)^2] dx' dx_n \\ &\leq C d_j^{2(m+1)} [\|p\|_{H^1(\omega_j)}^2 + \dots + d_j^{2(k-1)} \|p\|_{H^k(\omega_j)}^2 + \dots + d_j^{2(m-1)} \|p\|_{H^m(\omega_j)}^2] \\ &\leq C d_j^{2(m+1)} \|p\|_{H^1(\omega_j)}^2, \end{aligned}$$

and this completes the proof of  $\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \leq C d_j^{m+1} \|p\|_{H^1(\omega_j)}$ .

The proof of  $\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} \leq C d_j^m \|p\|_{H^1(\omega_j)}$  is reduced to the previous inequality as follows. First, from the inverse inequality  $d_j \|p\|_{H^1(\omega_j)} \leq C \|p\|_{L^2(\omega_j)}$ , we obtain

$$(38) \quad \|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \leq C d_j^m \|p\|_{L^2(\omega_j)}.$$

It is then enough to show that

$$(39) \quad \|\partial_k(p \circ \tilde{g}_j^{-1}) - \partial_k(p \circ \tilde{q}_j^{-1})\|_{L^2(\omega_j)} \leq C d_j^m \|p\|_{H^1(\omega_j)},$$

for all  $k = 1, 2, \dots, n$ .

The case  $k = n$  is easier, so we shall treat only the case when  $1 \leq k \leq n - 1$ . A Taylor expansion with respect to the  $x_n$ -variable gives

$$\begin{aligned}\partial_k(p \circ \tilde{g}_j^{-1})(x', x_n) &= \partial_k(p(x', x_n - g_j(x')))) \\ &= (\partial_k p)(x', x_n - g_j(x')) - \partial_k g_j(x')(\partial_n p)(x', x_n - g_j(x'))\end{aligned}$$

and

$$\begin{aligned}\partial_k(p \circ \tilde{q}_j^{-1})(x', x_n) &= \partial_k(p(x', x_n - q_j(x')))) \\ &= (\partial_k p)(x', x_n - q_j(x')) - \partial_k q_j(x')(\partial_n p)(x', x_n - q_j(x'))\end{aligned}$$

Equation (39) then follows from Equation (38) and from the estimates  $q_j(x') = g_j(x') + O(d_j^{m+1})$ ,  $\partial_k q_j(x') = \partial_k g_j(x') + O(d_j^m)$  and  $|g_j(x')| \leq d_j$  for  $(x', x_n) \in \omega_j$  (see Equation (26) and Condition C).  $\square$

*Remark 5.2.* Let us observe that Condition A( $h_\mu$ ) was used implicitly in the proof of Lemma 5.1 when we used the inverse estimates  $d_j^{k-1} \|p\|_{H^k(\omega_j)} \leq C \|p\|_{H^1(\omega_j)}$ .

*Remark 5.3.* If (26) is replaced by the more restrictive condition  $|\partial^\alpha(g_j - q_j)| \leq C d_j^{m+1-|\alpha|}$ , for all  $|\alpha| \leq m+1$ , then the result of the above lemma can be extended as follows: For any polynomial  $p$  of order  $m$ , we have

$$\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^s(\omega_j)} \leq C d_j^{m+1-s} \|p\|_{H^1(\omega_j)}, \quad s = 0, \dots, m+1,$$

where  $C$  is a constant independent of  $p$ ,  $\mu$ ,  $j$ , and  $s$ .

**Corollary 5.4.** *Let  $p \in \mathcal{P}_m$ , then*

$$\|\phi_j(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1})\|_{H^1(\omega_j)} \leq C d_j^m \|p\|_{H^1(\omega_j)}.$$

*If  $p \in \mathcal{P}_m$  also vanishes on  $\{x_n = 0\}$  then we have*

$$\|\phi_j(p \circ \tilde{q}_j^{-1})\|_{L^2(\partial\Omega)} \leq C d_j^m \|p\|_{H^1(\omega_j^*)}.$$

*Here  $C$  is a constant independent of  $p$ ,  $\mu$ , and  $j$ .*

*Proof.* Using Lemma 5.1 and Assumption B, we obtain

$$\begin{aligned}\|\phi_j(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1})\|_{H^1(\omega_j)} &\leq \|\phi_j\|_{L^\infty(\omega_j)} \|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} \\ &\quad + \left(\sum_{i=1}^n \|\partial_i \phi_j\|_{L^\infty(\omega_j)}\right) \|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \\ &\leq C d_j^m \|p\|_{H^1(\omega_j)} + C d_j^{-1} d_j^{m+1} \|p\|_{H^1(\omega_j)} \leq C d_j^m \|p\|_{H^1(\omega_j)}.\end{aligned}$$

The last part follows from the first part of this corollary, which we have already proved, and from the fact that  $\phi_j(p \circ \tilde{g}_j^{-1}) = 0$  on  $\partial\Omega$ . Indeed,

$$\begin{aligned}\|\phi_j(p \circ \tilde{q}_j^{-1})\|_{L^2(\partial\Omega)} &= \|\phi_j(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1})\|_{L^2(\partial\Omega)} \\ &\leq C \|\phi_j(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1})\|_{H^1(\Omega)} = C \|\phi_j(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1})\|_{H^1(\omega_j)} \\ &\leq C d_j^m \|p\|_{H^1(\omega_j)} \leq C d_j^m \|p\|_{H^1(\omega_j^*)}\end{aligned}$$

The proof is now complete.  $\square$

We are ready now to prove that Assumption 1 is satisfied by the sequence of GFEM-spaces  $S_\mu$  introduced in Subsection 4.2.

**Proposition 5.5.** *Let  $S_\mu$  be the sequence of GFEM-spaces defined by data  $\Sigma_\mu$  (Equation (24)) satisfying conditions A( $h_\mu$ ), B, C, and D. Then the sequence  $S_\mu$  satisfies Assumption 1.*

*Proof.* Let  $w_j \in \Psi_j^\mu$  and  $w = \sum \phi_j w_j \in S_\mu$ . Since we are interested in evaluating  $w$  at  $\partial\Omega$ , we can assume that only the terms corresponding to  $j$  for which  $\bar{\omega}_j \cap \partial\Omega \neq \emptyset$  appear in the sum. Then  $w_j = p_j \circ \tilde{q}_j^{-1}$ , for some polynomials  $p_j \in \mathcal{P}_m$  vanishing on  $\{x_n = 0\}$ . Hence Lemma 4.3 and Corollary 5.4 give

$$\begin{aligned} \|w\|_{H^{1/2}(\partial\Omega)}^2 &\leq C \sum_j \|\phi_j w_j\|_{L^2(\partial\Omega)}^2 = C \sum_j \|\phi_j(p_j \circ \tilde{q}_j^{-1})\|_{L^2(\partial\Omega)}^2 \\ &\leq C \sum_j d_j^{2m} \|p_j\|_{H^1(\omega_j^*)}^2 \leq Ch_\mu^{2m} \sum_j \|p_j\|_{H^1(\omega_j^*)}^2 \leq Ch_\mu^{2m} \sum_j \|w_j\|_{H^1(\omega_j^*)}^2. \end{aligned}$$

By Condition D,  $\sum_j \|w_j\|_{H^1(\omega_j^*)}^2 = \|w\|_{H^1(\cup \omega_j^*)}^2$ . Therefore,

$$\|w\|_{L^2(\partial\Omega)}^2 \leq Ch_\mu^{2m} \|w\|_{H^1(\cup \omega_j^*)}^2 \leq Ch_\mu^{2m} \|w\|_{H^1(\Omega)}^2.$$

Assumption 1 is hence satisfied by taking square roots.  $\square$

*Remark 5.6.* Condition D is only needed in the proof of Proposition 5.5. Although one can prove that

$$(40) \quad \sum_j \|w_j\|_{H^1(\omega_j^*)}^2 \leq C \|w\|_{H^1(\Omega)}^2$$

(by using norm equivalence in finite dimensional spaces), one can not bypass Condition D because the constant  $C$  in (40) depends on  $\mu$ . To remove this dependence, one would have to impose additional and/or different conditions on the partition of unity.

The proof that the sequence  $S_\mu$  also satisfies Assumption 2 is also based on the above lemma and on the following result. Recall that the local approximation spaces  $\Psi_j$  and  $\tilde{\Psi}_j^\mu$  were defined in Subsection 4.2.

**Lemma 5.7.** *Let  $u \in H^{m+1}(\omega_j)$ . Then there exists a polynomial  $w \in \tilde{\Psi}_j^\mu$  such that  $\|u - w\|_{H^1(\omega_j)} \leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)}$  and  $\|u - w\|_{L^2(\omega_j)} \leq Cd_j^{m+1} \|u\|_{H^{m+1}(\omega_j)}$  for a constant  $C$  independent of  $u$ ,  $\mu$ , and  $j$ . If  $u = 0$  on  $\bar{\omega}_j \cap \partial\Omega$ , then we can choose  $w \in \Psi_j^\mu$ .*

*Proof.* We are especially interested in the case when  $u = 0$  on  $\bar{\omega}_j \cap \partial\Omega$ , so we shall deal with this case in detail. The other one is proved in exactly the same way.

Let us consider  $v = u \circ \tilde{g}_j$ . Since  $\tilde{g}_j$  maps  $\mathbb{R}^{n-1} = \{x_n = 0\}$  to a surface containing  $\bar{\omega}_j \cap \partial\Omega$ , we obtain that  $v = 0$  on  $\mathbb{R}^{n-1}$ . For  $h_\mu$  small enough, we can assume that  $\tilde{g}_j^{-1}(\omega_j)$  lies on one side of  $\mathbb{R}^{n-1}$ . Let  $U$  be the union of the closure of  $\tilde{g}_j^{-1}(\omega_j)$  and of its symmetric subset with respect to  $\mathbb{R}^{n-1}$ . Define  $v_1 \in H^1(U)$  to be the odd extension of  $v$  (odd with respect to the reflection about the subspace  $\mathbb{R}^{n-1}$ ). Let  $p_1$  be the projection of  $v_1$  onto the subspace  $\mathcal{P}_m$  of polynomials of degree  $m$  on  $U$ . This projection maps even functions to even functions and odd functions to odd functions. Hence  $p_1$  is also odd. In particular,  $p_1 = 0$  on  $\mathbb{R}^{n-1}$ . We also know from standard approximation results [16] that

$$\|v_1 - p_1\|_{H^1(U)} \leq Cd_j^m \|v_1\|_{H^{m+1}(U)}.$$

Then

$$\|u - p_1 \circ \tilde{g}_j^{-1}\|_{H^1(\omega_j)} \leq C \|v_1 - p_1\|_{H^1(U)} \leq Cd_j^m \|v_1\|_{H^{m+1}(U)} \leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)}.$$

Let  $w = p_1 \circ \tilde{q}_j^{-1}$ . The lemma follows from

$$\begin{aligned} \|u - w\|_{H^1(\omega_j)} &\leq \|u - p_1 \circ \tilde{g}_j^{-1}\|_{H^1(\omega_j)} + \|p_1 \circ \tilde{g}_j^{-1} - p_1 \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} \\ &\leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)} + Cd_j^m \|p_1\|_{H^1(\omega_j)} \leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)}, \end{aligned}$$

where we have also used Lemma 5.1.

To prove the relation  $\|u - w\|_{L^2(\omega_j)} \leq Cd_j^{m+1} \|u\|_{H^{m+1}(\omega_j)}$ , we first notice that Poincaré's inequality gives

$$\|v_1 - p_1\|_{L^2(U)} \leq Cd_j \|v_1 - p_1\|_{H^1(U)} \leq Cd_j^{m+1} \|v_1\|_{H^{m+1}(U)}.$$

The rest is exactly the same.  $\square$

We are ready now to prove Assumption 2. See [4], section 6.1, and [9] for related results.

**Proposition 5.8.** *The sequence of GFEM spaces  $S_\mu$  satisfies Assumption 2.*

*Proof.* We proceed as in [9], Theorem 3.2. Let  $u \in H^{m+1}(\Omega)$ . If  $\bar{\omega}_j$  does not intersect  $\partial\Omega$ , we define  $w_j \in \Psi_j = \mathcal{P}_m$  to be the orthogonal projection of  $u$  onto  $\mathcal{P}_m$  in  $H^1(\omega_j)$ . Otherwise, we define  $w_j \in \Psi_j$  using Lemma 5.7. Then let  $w = \sum_j \phi_j w_j$ . By using Lemma 5.7, the definition of the local approximation spaces  $\Psi_j$  (Condition C), and the bounds on  $\|\nabla \phi_j\|_{L^\infty}$  (Condition B), we obtain

$$\begin{aligned} |u - w|_{H^1(\Omega)} &= \left| \sum_j \phi_j (u - w_j) \right|_{H^1(\Omega)} \\ &\leq \sum_j (\|\phi_j\|_{L^\infty} |u - w_j|_{H^1(\omega_j)} + \|\nabla \phi_j\|_{L^\infty} \|u - w_j\|_{L^2(\omega_j)}) \\ &\leq \sum_j (Cd_j^m \|u\|_{H^{m+1}(\omega_j)} + Cd_j^{-1} d_j^{m+1} \|u\|_{H^{m+1}(\omega_j)}) \leq C\kappa h_\mu^m \|u\|_{H^{m+1}(\Omega)}. \end{aligned}$$

This completes the result.  $\square$

Next, we will be looking for a sequence  $G_k$  of *approximate extensions of  $g$*  in  $\tilde{S}_k$ . Recall that the spaces  $\tilde{S}_k \supset S_k$  were defined in Equation (29) and are variants of the spaces  $S_k$  that are not required to satisfy, even approximately, the boundary conditions. The construction of such a sequence  $G_k$  of approximate extension as well as the analysis of the resulting method are the main results of this subsection. Other methods for constructing  $G_k$  are certainly possible.

We now check that it is possible to choose  $G_k \in \tilde{S}_k$  satisfying Assumption 3. We follow the method in [3].

**Proposition 5.9.** *There exist continuous linear maps  $I_k : H^{m+1}(\Omega) \rightarrow \tilde{S}_k$ , such that*

$$(41) \quad |u - I_k(u)|_{H^r(\Omega)} \leq Ch_k^{m+1-r} \|u\|_{H^{m+1}(\Omega)},$$

for  $r = 0$  and  $r = 1$ .

*Proof.* For  $u \in H^{m+1}(\Omega)$  and  $j$  fixed, let  $v = u \circ \tilde{g}_j$ . The Taylor polynomial of degree  $m$  of  $v$  averaged over  $\tilde{g}_j^{-1}(\omega_j)$  is given by

$$(42) \quad P_j(x) := Q_j^m v(x) = \int_{\tilde{g}_j^{-1}(\omega_j)} Q_{y,v,n}(x) \Phi_j(y) dy,$$

where

$$Q_{y,v,n}(x) = v(y) + \sum_{i=1}^n \partial_i v(y)(x_i - y_i) + \dots + \sum_{|\alpha|=m} \frac{v^{(\alpha)}(y)}{\alpha!} (x-y)^\alpha, \quad \alpha! = \alpha_1! \dots \alpha_n!,$$

is the Taylor polynomial of  $v$  at  $y$  of degree  $m$  and  $\Phi_j \in C_c^\infty(\tilde{g}^{-1}(\omega_j))$  is a function with integral 1. Then, by the Bramble–Hilbert Lemma, we have

$$(43) \quad |v - P_j|_{H^s(\tilde{g}^{-1}(\omega_j))} \leq Ch_k^{m+1-s} |v|_{H^{m+1}(\tilde{g}^{-1}(\omega_j))}, \quad \text{for all } 0 \leq s \leq m+1.$$

Consider  $w_j := P_j \circ \tilde{q}_j^{-1} \in \tilde{\Psi}_j$ . Let  $w := \sum_j \phi_j w_j$ . Then,

$$\begin{aligned} (44) \quad |u - w|_{H^r(\Omega)}^2 &\leq C \sum_j |\phi_j(u - w)|_{H^r(\Omega)}^2 \leq C \sum_j |\phi_j(u - w)|_{H^r(\omega_j)}^2 \\ &\leq C \sum_j \sum_{i=0}^r |\phi_j|_{W^{i,\infty}(\omega_j)}^2 |u - w_j|_{H^{r-i}(\omega_j)}^2 \\ &\leq C \sum_j \sum_{i=0}^r |\phi_j|_{W^{i,\infty}(\omega_j)}^2 [|u - P_j \circ \tilde{g}_j^{-1}|_{H^{r-i}(\omega_j)}^2 + |P_j \circ \tilde{g}_j^{-1} - P_j \circ \tilde{q}_j^{-1}|_{H^{r-i}(\omega_j)}^2]. \end{aligned}$$

By changing variables and (43), we obtain

$$\begin{aligned} (45) \quad |u - P_j \circ \tilde{g}_j^{-1}|_{H^{r-i}(\omega_j)}^2 &= |v \circ \tilde{g}_j^{-1} - P_j \circ \tilde{g}_j^{-1}|_{H^{r-i}(\omega_j)}^2 \leq C |v - P_j|_{H^{r-i}(\tilde{g}_j^{-1}(\omega_j))}^2 \\ &\leq Ch_k^{2(m+1-r+i)} |v|_{H^{m+1}(\tilde{g}^{-1}(\omega_j))}^2 = Ch_k^{2(m+1-r+i)} |u \circ \tilde{g}_j|_{H^{m+1}(\tilde{g}^{-1}(\omega_j))}^2 \\ &\leq Ch_k^{2(m+1-r+i)} |u|_{H^{m+1}(\omega_j)}^2. \end{aligned}$$

Also, from Lemma 5.1 and the definition (42) of  $P_j$ , we have

$$\begin{aligned} (46) \quad |P_j \circ \tilde{g}_j^{-1} - P_j \circ \tilde{q}_j^{-1}|_{H^{r-i}(\omega_j)}^2 &\leq Ch_k^{2(m+1-r+i)} \|P_j\|_{H^1(\omega_j)}^2 \\ &\leq Ch_k^{2(m+1-r+i)} \|u\|_{H^{m+1}(\omega_j)}^2. \end{aligned}$$

From (44), (45), (46), and Condition B, it follows that

$$\begin{aligned} (47) \quad |u - w|_{H^r(\Omega)}^2 &\leq C \sum_j \sum_{i=0}^r h_k^{-2i} [h_k^{2(m+1-r+i)} |u|_{H^{m+1}(\omega_j)}^2 + h_k^{2(m+1-r+i)} \|u\|_{H^1(\omega_j)}^2] \\ &\leq Ch_k^{2(m+1-r)} \sum_j \|u\|_{H^{m+1}(\omega_j)}^2 \leq C \kappa h_k^{2(m+1-r)} \|u\|_{H^{m+1}(\Omega)}^2, \end{aligned}$$

for all  $0 \leq r \leq m+1$ .

Define  $I_k(u) := w$ . Clearly  $I_k$  is a linear map from  $H^{m+1}(\Omega)$  to  $\tilde{S}_k$  which satisfies (41). This ends the proof of the proposition.  $\square$

*Remark 5.10.* If we assume the stronger condition stated in Remark 5.3 on the  $m$ -degree polynomial  $q_j$ , then the conclusion of Proposition 5.9 is valid for  $0 \leq r \leq m+1$  (the proof being exactly the same).

From this proposition we obtain right away the Assumption 3.

**Proposition 5.11.** *For any  $g \in H^{m+1/2}(\partial\Omega)$  there exists a sequence  $G_k \in \tilde{S}_k$  satisfying Assumption 3.*

*Proof.* Indeed, let us chose  $G \in H^{m+1}(\Omega)$  that extends  $g$  to the interior and satisfies  $\|G\|_{H^{m+1}(\Omega)} \leq C\|g\|_{H^{m+1/2}(\partial\Omega)}$ , with  $C$  independent of  $g$ . Then choose  $G_k = I_k(G)$ , with  $I_k$  as in Proposition 5.9.  $\square$

## 6. INTERIOR NUMERICAL APPROXIMATION

In this section, we construct a sequence of approximations  $u_k \in \tilde{S}_k$  of the solution  $u$  of the distributional boundary value problem  $\Delta u = 0$  in  $\Omega$ ,  $u = g$  on  $\partial\Omega$ , with  $g \in H^{1/2-s}(\partial\Omega)$ , and we prove interior estimates for the error  $u - u_k$ . The sequence of spaces  $\tilde{S}_k$  is the sequence of GFEM spaces constructed in Section 4 and hence it satisfies the Assumptions 2 and 3, by the results of Section 5. In particular,  $S_k \subset \tilde{S}_k$ . We need to consider GFEM spaces instead of the more general framework of the first part because we need the interior approximation result of [9] recalled below.

In this section,  $s \in \mathbb{Z}_+$  is *fixed*. Our results mirror the ones in [9], where the Neumann problem was considered. The approach is different however, in part because it is more involved to construct finite element approximations of the solution  $u$  in the case of the Dirichlet boundary conditions.

Our approach is to first approximate  $g$  with a sequence  $G_k$  of functions. Then each of the approximate equations  $\Delta v_k = 0$ ,  $v_k = G_k$  at the boundary, is solved approximately using the results of the previous sections. This yields, for any  $k$ , a sequence  $(v_k)_\mu$  with  $(v_k)_\mu \in \tilde{S}_\mu$ . The sequence of approximations is then  $u_\mu := (v_\mu)_\mu \in \tilde{S}_\mu$ . Our approach is thus similar to that of Section 2.

Recall the spaces  $\tilde{S}_\mu$  and  $S_\mu$  introduced in Assumptions 1, 2, and 3. Let us also recall that  $S_\mu \subset \tilde{S}_\mu$ . The definition of the space  $S_\mu$  is slightly different from the one in [9]; however, the difference is only in the local approximation spaces at the boundary, and hence this does not affect the spaces  $S_\mu^<(\Omega) := S_\mu \cap \mathcal{C}_c^\infty(\Omega)$ . Therefore, it can be shown that Theorem 3.12 from [9] is still valid. Namely, we have the following:

**Theorem 6.1.** *Let  $A \Subset B \subset \Omega$  be open subsets. Then there exists  $C > 0$  with the following property. If  $u \in H^1(\Omega)$  and  $u_\mu \in \tilde{S}_\mu$  are such that  $B(u - u_\mu, \chi) = 0$  for all  $\chi \in S_\mu^<(\Omega)$ , then for  $h_\mu$  small enough,*

$$\|u - u_\mu\|_{H^1(A)} \leq C \left( \inf_{\chi \in S_\mu} \|u - \chi\|_{H^1(B)} + \|u - u_\mu\|_{H^{-m}(B)} \right).$$

*The constant  $C$  depends only on  $A$ ,  $B$ , and the structural constants, but not on  $\mu \in \mathbb{Z}_+$ .*

This result is the version for the Generalized Finite Element of a basic result by Nitsche and Schatz [29, Theorem 5.1]. See also [31, Theorem 9.2]. The above result is the reason why we work in this section in the framework of the Generalized Finite Element Method and not in the abstract setting of the first part.

Recall that  $s$  is fixed in this section. We otherwise use the notation of the previous sections. We shall need the following property of the spaces  $\tilde{S}_j$ .

- *The low regularity approximate extension property:* There exists a constant  $C > 0$  such that, for any  $g \in H^{1/2-s}(\partial\Omega)$ , we can find a sequence  $G_j \in \tilde{S}_j$  satisfying  $\|G_j|_{\partial\Omega} - g\|_{H^{1/2-s-\gamma}(\partial\Omega)} \leq Ch_j^\gamma \|g\|_{H^{1/2-s}(\partial\Omega)}$  for all  $0 \leq \gamma \leq m$  and  $\|G_j\|_{H^1(\Omega)} \leq Ch_j^{-s} \|g\|_{H^{1/2-s}(\partial\Omega)}$ .

**Lemma 6.2.** *Assumptions 2 and 3 imply the low regularity approximate extension property. In particular, the spaces  $\tilde{S}_j$  satisfy the low regularity approximate extension property.*

*Proof.* Let  $g \in H^{1/2-s}(\partial\Omega)$ . As in [30], we shall consider for each  $H > 0$  functions  $G_H \in C^\infty(\bar{\Omega})$  such that

$$\begin{aligned} \|g - G_H\|_{H^{1/2-s-\gamma}(\partial\Omega)} &\leq CH^\gamma \|g\|_{H^{1/2-s}(\partial\Omega)} \\ \|G_H\|_{H^1(\Omega)} &\leq CH^{-s} \|g\|_{H^{1/2-s}(\partial\Omega)}, \end{aligned}$$

with constants independent of  $g$ . On  $\mathbb{R}^n$ , we can choose  $G_H = \chi_H * g$ , with  $\chi_H(x) = H^{-n} \chi(x/H)$  for a suitable chosen  $\chi \in C_c^\infty(\mathbb{R}^n)$ . In general, this procedure can be localized as in [15].

Let us then choose  $H = h_j$  and  $G_j \in \tilde{S}_j$  to have the following projection property

$$B(G_H - G_j, \chi) = 0, \quad \chi \in \tilde{S}_j, \quad \text{and} \quad \int_{\partial\Omega} (G_H - G_j) dS(x) = 0.$$

Then  $\|G_j\|_{H^1(\Omega)} \leq \|G_H\|_{H^1(\Omega)} \leq Ch_j^{-s} \|g\|_{H^{1/2-s}(\partial\Omega)}$ . To prove our result, it is enough to show that the restrictions to the boundary  $\partial\Omega$  satisfy

$$(48) \quad \|G_j - G_H\|_{H^{1/2-l}(\partial\Omega)} \leq Ch_j^l \|G_H\|_{H^1(\Omega)},$$

for any  $l \geq 0$  (for our result we then take  $l = s + \gamma$ ).

We shall proceed as in [3]. Let  $w \in H^{l-1/2}(\partial\Omega)$  with  $\int_{\partial\Omega} w dS(x) = 0$ . We let  $W \in H^{l+1}(\Omega)$  be the unique solution with mean zero of the Neumann problem  $\Delta W = 0$ ,  $\partial_\nu W = w$ . Then  $\|W\|_{H^{l+1}(\Omega)} \leq C \|w\|_{H^{l-1/2}(\partial\Omega)}$  and hence Assumption 2 (approximation property) gives that there exists  $\chi \in \tilde{S}_j$  such that  $|W - \chi|_{H^1(\Omega)} \leq Ch_j^l \|W\|_{H^{l+1}(\Omega)} \leq Ch_j^l \|w\|_{H^{l-1/2}(\partial\Omega)}$ . We obtain

$$\begin{aligned} \|G_j - G_H\|_{H^{1/2-l}(\partial\Omega)} &= \sup_{w \neq 0} \frac{\langle G_j - G_H, w \rangle_{\partial\Omega}}{\|w\|_{H^{l-1/2}(\partial\Omega)}} = \sup_{w \neq 0} \frac{B(G_j - G_H, W - \chi)}{\|w\|_{H^{l-1/2}(\partial\Omega)}} \\ &\leq C |G_j - G_H|_{H^1(\Omega)} \sup_{w \neq 0} \frac{|W - \chi|_{H^1(\Omega)}}{\|w\|_{H^{l-1/2}(\partial\Omega)}} \leq Ch_j^l \|G_H\|_{H^1(\Omega)}. \end{aligned}$$

In view of the fact that  $\int_{\partial\Omega} (G_j - G_H) dS(x) = 0$ , this proves Equation (48) and hence completes the proof of the Lemma.  $\square$

We are now ready to prove a result on the interior approximation properties for boundary value problems with low regularity (*i.e.*, distributional) boundary data.

**Theorem 6.3.** *Let  $S_\mu \subset \tilde{S}_\mu$  be our sequences of GFEM spaces. Also, let  $g \in H^{1/2-s}(\partial\Omega)$ ,  $1 \leq s \leq m-1$ ,  $A \Subset \Omega$ , and  $G_j$  be as in the low regularity approximate extension property. We define  $(v_j)_\mu$  to be the discrete solution of the equation  $\Delta u = 0$ ,  $u = G_j$  at the boundary, as defined in Equation (54) and  $u_\mu := (v_\mu)_\mu$ . Then*

$$\|u - u_\mu\|_{H^1(A)} \leq Ch_\mu^{m-s-1} \|g\|_{H^{1/2-s}(\partial\Omega)}.$$

*Proof.* We now proceed as in Section 2. Let us denote by  $w_k$  the solution of

$$(49) \quad -\Delta w_k = \Delta G_k \in L^2(\Omega) \quad \text{in } \Omega \quad \text{and} \quad w_k = 0 \quad \text{on } \partial\Omega,$$

Then we let  $v_k := w_k + G_k$ , which will satisfy  $\Delta v_k = 0$  in  $\Omega$ ,  $v_k = G_k$  on  $\partial\Omega$ .

Lemma 6.2 shows that the low regularity approximate extension property is satisfied. Theorem 0.3 and the low regularity approximate extension property then give

$$(50) \quad \|u - v_k\|_{H^{1-s-\gamma}(\Omega)} \leq C\|g - G_k\|_{H^{1/2-s-\gamma}(\partial\Omega)} \leq Ch_k^\gamma \|g\|_{H^{1/2-s}(\partial\Omega)}.$$

Let  $B$  be an open set such that  $A \Subset B \Subset \Omega$  and  $t$  be a parameter. Then for any harmonic function  $\phi \in \mathcal{C}^\infty(\Omega)$ , there exists a constant  $C$  such that  $\|\phi\|_{H^t(B)} \leq C\|\phi\|_{H^{1-s-\gamma}(\Omega)}$ . Taking  $\phi := u - v_k$ ,  $t = 1$ , and using also Equation (50), we obtain

$$(51) \quad \|u - v_k\|_{H^1(B)} \leq C\|u - v_k\|_{H^{1-s-\gamma}(\Omega)} \leq Ch_k^\gamma \|g\|_{H^{1/2-s}(\partial\Omega)}.$$

By taking  $\phi = v_k$ , we obtain

$$(52) \quad \|v_k\|_{H^{m+1}(B)} \leq C\|v_k\|_{H^{1-s}(\Omega)} \leq C\|G_k\|_{H^{1/2-s}(\partial\Omega)} \leq C\|g\|_{H^{1/2-s}(\partial\Omega)}.$$

Also, let us denote by  $(w_k)_\mu \in S_\mu$  the discrete solution of the problem (49) Namely,

$$(53) \quad B((w_k)_\mu, \chi) = (\Delta G_k, \chi), \quad \chi \in S_\mu.$$

This is nothing but Equation (2) for  $f = 0$ . Let

$$(54) \quad (v_k)_\mu := (w_k)_\mu + G_k.$$

Then Theorem 1.6 gives

$$(55) \quad \|v_k - (v_k)_\mu\|_{H^{-l}(\Omega)} \leq Ch_\mu^{p+l+1} \|\Delta G_k\|_{H^{p-1}(\Omega)} \leq Ch_\mu^{p+l+1} \|G_k\|_{H^{p+1}(\Omega)},$$

for  $p + l + 1 \leq m$ .

Let us now take  $p = 0$  and  $l = s + \gamma - 1$ , which satisfy  $s + \gamma = p + l + 1 \leq m$ . Then

$$(56) \quad \|v_k - (v_k)_\mu\|_{H^{1-s-\gamma}(\Omega)} \leq Ch_\mu^{s+\gamma} \|G_k\|_{H^1(\Omega)} \leq Ch_\mu^{s+\gamma} h_k^{-s} \|g\|_{H^{1/2-s}(\partial\Omega)}.$$

In particular, for  $k = \mu$ , we obtain

$$(57) \quad \|v_\mu - (v_\mu)_\mu\|_{H^{1-s-\gamma}(\Omega)} \leq Ch_\mu^\gamma \|g\|_{H^{1/2-s}(\partial\Omega)}.$$

Since  $B(v_\mu - (v_\mu)_\mu, \chi) = B(w_\mu - (w_\mu)_\mu, \chi) = 0$  for all  $\chi \in S_\mu$ , Theorem 6.1 then gives

$$(58) \quad \|v_\mu - (v_\mu)_\mu\|_{H^1(A)} \leq C \left( \inf_{\chi \in S_\mu} \|v_\mu - \chi\|_{H^1(B)} + \|v_\mu - (v_\mu)_\mu\|_{H^{-m}(B)} \right) \\ \leq C(h_\mu^m + h_\mu^\gamma) \|g\|_{H^{1/2-s}(\partial\Omega)} \leq Ch_\mu^\gamma \|g\|_{H^{1/2-s}(\partial\Omega)}$$

where the last two terms were estimated using Equations (52) and (57).

Equations (51) and (57) give for  $s + \gamma + 1 = m$  that

$$\|u - (v_\mu)_\mu\|_{H^1(A)} \leq \|u - v_\mu\|_{H^1(A)} + \|v_\mu - (v_\mu)_\mu\|_{H^1(A)} \leq Ch_\mu^\gamma \|g\|_{H^{1/2-s}(\partial\Omega)}$$

The proof is complete.  $\square$

## 7. COMMENTS AND FURTHER PROBLEMS

In spite of all the differences in assumptions and definitions between [11, 28] and our paper, the main issue seems to be providing simple examples of spaces satisfying the various assumptions used in these papers. For instance, it would be interesting to provide other examples of spaces  $S_\mu$  satisfying Assumptions 1 and 2. It would also be interesting to see if a modification of the uniform partition of unity can give, by restriction, spaces  $S_\mu$  satisfying these Assumptions. Finally, it would be important to integrate our results with the issues arising from numerical integration and to provide explicit numerical examples testing our results. In fact,

related numerical tests together with some theoretical results can be found, for example, in [5, 20, 30].

**7.1. Comments on the approximate boundary conditions.** For most of our results, it is enough to assume

$$\|v_\mu\|_{L^2(\partial\Omega)} \leq Ch_\mu^m \|v_\mu\|_{H^1(\Omega)} \quad \text{for any } v_\mu \in S_\mu.$$

However, one then has to replace  $\|u\|_{H^1(\Omega)}$  with  $\|u\|_{H^2(\Omega)}$  in Lemmas 1.4 and 1.5. Also, the proof of Theorem 1.6 for  $p = 0$  requires the full strength of Assumption 1. The case  $p = 0$  is the one needed for the results of Section 6, and that is the only real reason for which we need the stronger form of Assumption 1.

A related problem is to construct other examples of spaces satisfying the interior estimates used in Section 6 as well as the general Assumptions 1, 2, and 3. Then Theorem 6.3 would be valid for these spaces as well.

## REFERENCES

- [1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton, N.J., 1965.
- [2] I. Babuška, *Error-bounds for finite element method*, Numer. Math., vol. **16**, pp. 322-333, 1971.
- [3] I. Babuška and A. K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In *The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*, pages 1–359. Academic Press, New York, 1972. With the collaboration of G. Fix and R. B. Kellogg.
- [4] I. Babuška, U. Banerjee, and J.E. Osborn, *Survey of meshless and generalized finite element methods: A unified approach*, Acta Numerica, pp. 1-125, 2003.
- [5] I. Babuška, G. Caloz, and J.E. Osborn, *Numerical implementation of GFEM*, in final preparation.
- [6] I. Babuška, G. Caloz, and J.E. Osborn, *Special finite element methods for a class of second order elliptic problems with rough coefficients*, SIAM J. Numer. Anal., vol **31**, no. 4, pp. 945-981, 1994.
- [7] I. Babuška, G. Caloz, and J. E. Osborn. *Generalized finite element methods. main ideas, results and perspectives*, Internat. J. Comp. Meth., 1(1):67–102, 2004.
- [8] I. Babuška and J.M. Melenk, *The Partition of Unity Method*, Internat. J. Num. Methods Engrg., vol. **40**, pp. 727-758, 1997.
- [9] I. Babuška, V. Nistor, *Interior Numerical Approximation of Boundary Value Problems with a Distributional Data*, Numerical Meth. PDEs **2** (2006), 79-113.
- [10] C. Bacuta, J. Chen, Y. Huang, J. Xu, and L. Zikatanov, *Partition of unity method on non-matching grids for the Stokes problem*, Journal of Numerical Mathematics **13** (2005), 157-169.
- [11] A. Berger, R. Scott, and G. Strang, *Approximate boundary conditions in the finite element method*, Symposia Mathematica, Academic Press, New York, 1972.
- [12] J.H. Bramble, J.A. Nitsche, *A Generalized Ritz–Least–Squares Method for Dirichlet Problems*, SIAM J. Numer. Anal, vol **10**, no. 1, pp. 81-93, 1973.
- [13] J.H. Bramble and A.H. Schatz, *Rayleigh–Ritz–Galerkin methods for Dirichlet’s problem using subspaces without boundary conditions*, Comm. Pure Appl. Math., vol. **23**, pp. 653-675, 1970.
- [14] J.H. Bramble and A.H. Schatz, *Least squares methods for  $2m^{\text{th}}$  order elliptic boundary-value problems*, Math. Comp., vol. **25**, pp. 1-32, 1971.
- [15] J.H. Bramble and A.H. Schatz, *Higher order local accuracy by averaging in the finite element method*, Math. Comp., vol. **31**, pp. 94–111, 1977.
- [16] S. Brenner and R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2002.
- [17] P. Ciarlet. Basic error estimates for elliptic problems. In *Handbook of numerical analysis, Vol. II*, Handb. Numer. Anal., II, pages 17–352. North-Holland, Amsterdam, 1991.
- [18] P.G. Ciarlet and P.A. Raviart, *The combined effect of curved boundaries and numerical integration in isoparametric finite element methods*, The mathematical foundations of the

- finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972), pp. 409–474. Academic Press, New York, 1972.
- [19] K. Dossou and M. Fontaine, *A high order isoparametric finite element method for the computation of waveguide modes*, Comput. Methods Appl. Mech. Engrg. **194** (2005), no. 6-8, 837–858.
  - [20] A. Duarte, D.-J. Kim, and D.M. Quaresma, *Arbitrarily smooth Generalized Finite Element approximations*, to appear in CMAME.
  - [21] E. Hernandez, *Approximation of the vibration modes of a plate coupled with a fluid by low-order isoparametric finite elements*, M2AN Math. Model. Numer. Anal. **38** (2004), no. 6, 1055–1070.
  - [22] Y. Huang and Jinchao Xu, *A conforming finite element method for overlapping and non-matching grids*, Mathematics of Computation **72** (2003), 1057–1066.
  - [23] M.-P. Lebaud, *Error estimate in an isoparametric finite element eigenvalue problem*, Math. Comp. **63** (1994), no. 207, 19–40.
  - [24] L.K. Li, *Approximate boundary condition and numerical integration in isoparametric finite element methods*, Proceedings of the China-France symposium on finite element methods (Beijing, 1982), 785–814, Science Press, Beijing, 1983.
  - [25] S. Li and Wing-Kam Liu, *Particles Methods*. Springer-Verlag, Berlin, 2004.
  - [26] J. M. Melenk and I. Babuška. The partition of unity finite element method: basic theory and applications. *Comput. Methods Appl. Mech. Engrg.*, 139(1-4):289–314, 1996.
  - [27] J.A. Nitsche, *A projection method for Dirichlet problems using subspaces with nearly zero boundary conditions* (unpublished manuscript).
  - [28] J.A. Nitsche, *On Dirichlet Problems Using Subspaces with Nearly Zero Boundary Conditions*, in “The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations,” edited by A.K. Aziz, Academic Press, pp. 603-627, 1972.
  - [29] J.A. Nitsche and A. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp. **28** (1974), 937–958.
  - [30] A. Solo, *Sharp estimates for Finite Element approximations to elliptic problems with Neumann data of low regularity*, Preprint 2006.
  - [31] L. Wahlbin, *Local behavior in finite element methods*, Handbook of numerical analysis, Vol. II, North-Holland, Amsterdam, 1991, 353–522.
  - [32] L.H. Wang, *On the error estimate for the isoparametric finite element method*, J. Comput. Math. **3** (1985), no. 3, 211–218.
  - [33] W.X. Wang and Y. Takao, *Isoparametric finite point method in computational mechanics*, Comput. Mech. **33** (2004), no. 6, 481–490.
  - [34] J. Xu and L. Zikatanov, *On Multigrid Methods for Generalized Finite Element Methods*, Lect. Notes Comput. Sci. Eng. **26** (2003), 401-418.

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